

ASPECTS OF COOPERATIVE GAME THEORY AND
ITS APPLICATIONS TO ECONOMICS

By

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Dedicated to

My Mother and the Memory of my Father.

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TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	iii
ABSTRACT	vi
CHAPTER	
ONE INTRODUCTION	1
1.1 General Background	1
1.2 Applications to Economics	5
1.3 A Brief Statement of the Problem	10
1.4 Mathematical Notation	11
TWO ECONOMIC ENVIRONMENTS WITH EXTERNALITIES AND PUBLIC GOODS AS COOPERATIVE GAMES	14
2.1 Introduction	14
2.2 Economic Environments	19
2.3 Consistent Joint Actions	27
2.4 Coalition Actions	34
2.5 The Cooperative Game $\Gamma(\underline{e})$	39
2.6 Strategies for Coalitions in a Game $\Gamma(\underline{e})$	46
2.7 The Cooperative Game $\tilde{\Gamma}(\underline{e})$	59
2.8 Coalition Production Economies	63
THREE CORE-CONCEPTS FOR AN ECONOMIC ENVIRONMENT $\underline{e}$	66
3.1 Introduction	66
3.2 Core-Concepts for a Game $\Gamma(\underline{e})$	69
3.3 The Core* of a Game $\Gamma(\underline{e})$	74
3.4 The α -Core of a Game $\Gamma(\underline{e})$	79
3.5 Core-Concepts for a Game $\tilde{\Gamma}(\underline{e})$	86
3.6 Comparisons Between the Core* and the α -Core of a Game	91
3.7 Externalities and Public Goods	92
FOUR INDIVIDUALLY DEFENSIBLE POSITIONS AND THE EFFECTIVE CORE FOR n -PERSON COOPERATIVE GAMES	98
4.1 Introduction	98
4.2 Notation and Preliminary Definitions	102
4.3 Individually Strongly Defensible Positions	105

	Page
4.4 Individually Defensible Positions	114
4.5 The Effective Core	123
4.6 The Bargaining Set $M_1^{(i)}$, the Kernel, and Individually Defensible Positions for a Game ...	136
FIVE INDIVIDUALLY DEFENSIBLE ALLOCATIONS FOR AN ECONOMY	141
5.1 Introduction	141
5.2 Individually Defensible Positions	143
5.3 Crowded Public Goods	152
SIX SUMMARY AND CONCLUDING REMARKS	155
6.1 Summary	155
6.2 Concluding Remarks	159
APPENDIX THE EFFECTIVE CORE FOR 3-PERSON GAMES I	162
REFERENCES	169
BIOGRAPHICAL SKETCH	172

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Models of economies are considered where individuals can form different coalition structures with respect to different economic activities. Each coalition, in such coalition structures, is allowed to undertake a coalition action with respect to the relevant economic activity. Economic activities, including the production of public goods, could be the source of externalities.

Cooperative games are defined, to represent conflict of interest in such economies, where coalition actions constitute conditional strategies for the respective coalitions, that is, strategies which may depend on agreements made by each member of a given coalition with players outside that coalition concerning other economic activities.

Two core-concepts are introduced relative to such cooperative games with the following characteristics. In the absence of non-coalition-specific externalities, those two core-concepts could be identical to each other. In the presence of a non-coalition-specific externality, such as, a pure public good, the two may be as diverge as to have one of them empty while the other is very large.

Additional solution concepts, but for classical cooperative games, are introduced by postulating that a payoff vector will be stable if each player can "defend," either in a strong or a weak sense, his corresponding payoff against those, concerted, actions of the other players that could affect him most adversely. The two solution concepts that emerge from this postulate are the set of individually strongly defensible positions and the set of individually defensible positions of a game.

Based on the hypothesis that certain solution concepts can be defined on the basis of a "security criterion" and a "maximum benefit criterion," and using Pareto optimality for the latter, we prove that (a) the set of Pareto optimal individually strongly defensible positions constitutes the (usual) core of a game, and (b) the set of Pareto optimal individually defensible positions represents a core-concept, the effective core, which is not empty for every classical game.

The foundations of a theory for individually defensible allocations in an economy are laid by applying the above concepts of stability to cooperative games defined relative to the aforementioned models of economies.

Additional results have been also obtained.

CHAPTER ONE INTRODUCTION

1.1 General Background

Games and Solutions

Two major problems have preoccupied Game Theory since its inception in the work of Von Neumann and Morgenstern [25].

1. The formulation of rigorous mathematical models capable of representing conflicts of interest that may exist in a variety of social, economic, political, or other situations.

2. Given that a mathematical model capable of representing conflict of interest has been formulated, what, if any, of all possible outcomes are most likely to occur.

As a solution to the first problem, a variety of games of strategy, most commonly referred to, simply, as games, have been introduced in the literature, with different games best fitted to represent and analyze different situations of conflict of interest. Hence, games may differ on type (e.g., cooperative; non-cooperative) descriptive form (e.g., extensive, normal, characteristic function, etc.) or some other characteristic (e.g., constant or non-constant sum, with or without side payments, etc.).

As a solution to the second problem, a number of solution concepts have been introduced in the literature. Solution concepts may depend both, on the type and the descriptive form of a game.

In general, and because games of strategy are designed to represent conflict of interest but not necessarily the particular process that may be followed for solving such conflicts, different solution concepts are defined on the basis of different sets of stability criteria. This could be interpreted as follows. Given a game of strategy, and without knowing the particular process followed for solving the underlying conflict(s) of interest, one postulates a number of criteria such that, if a process, as above, could lead to an outcome that satisfies those criteria then, that process will end there. Outcomes that satisfy the postulated set of criteria can be considered stable in this sense, and they comprise the relevant solution concept of the relevant game. It is in this sense, then, that solution concepts could determine the likelihood for the occurrence of the various outcomes possible in a game. Apparently, based on different sets of stability criteria one could define different solution concepts for a given game.

Cooperative Games

There are situations of conflict of interest where certain benefits may be realized only through the cooperation of the interested parties. On the other hand, and as a matter of strategy, to cooperate with others may increase the bargaining power of an individual. Thus, groups of people with some common interests may, under certain circumstances, get together and act as a single unit, a coalition. This represents particular cases of conflict of interest that extends between groups which, for a variety of reasons may compete against each other. The games that deal with these kinds of conflict of interest belong to the class of cooperative games.

There are two aspects of cooperative games that we would like to focus our attention here.

1. The descriptive form of a cooperative game.
2. The solution concepts to such games.

In general, and depending on the complexity of the situation, to use a game in one descriptive form, rather than another, may simplify considerably the analysis. In particular, and because in a cooperative game one is concerned with the power of coalitions, it has been an established procedure in the literature to use the characteristic function form of that game.

This form requires the specification of the set of outcomes for which each coalition of players is effective in some well defined sense (see, e.g., Luce and Raiffa [19, p. 234], Aumann and Peleg [5], and Aumann [1]), that is, it requires the specification of a characteristic function. Now, and unless a characteristic function is specified from the outset, this function must be derived from some other, more basic, form of the game, such as the normal form. This requires a number of assumptions for determining the conditions under which coalitions are effective for the various outcomes. An implication of this is that (a) the characteristic function form of a game, so derived, may not be unique, i.e., the sense in which coalitions are effective for certain outcomes may vary, [1], [5], and (b) certain outcomes which are possible under one derivation may not be possible under another (see for example Aumann [3, p. 20]).

In [32], R. Rosenthal has presented two examples in order to demonstrate that the derivation of the characteristic function from the normal form of a game, under a variety of assumptions, may be inadequate

to deal with certain "features of cooperative decision situations." As such, he has listed: (a) the possibility that some strategies in the normal form of a game may have some conditional feature in them which cannot be reflected in a characteristic function, and (b) the restricted view about the threat possibilities that the characteristic function takes.

In addressing questions of stability for the outcomes of any given cooperative game, one particular solution concept, the core of a game (also referred to as the α -core in [1] and [5]) has played a prominent role in the literature. Briefly speaking, and using group rationality as a stability criterion, an outcome will belong to the core of a cooperative game, if no group of players, acting unilaterally, could depart from that outcome and, by so doing, improve the position of at least one member without making any member worse off.

Although, one has little to argue against the stability properties of outcomes in the core of a game, the fact is that many cooperative games may have an empty core. Hence, group rationality has been considered by many as a very strong criterion for stability. Alternative solution concepts to cooperative games, then, have been introduced in the literature which do not require stability criteria as strong as those of the core. To mention a few, we can start out with some of the immediate generalizations of the core such as, the β -core, Aumann and Peleg [5], Aumann [1], a number of quasi-cores including the ϵ -core, Shapley and Shubik [38], and the least core, Maschler, Peleg, and Shapley [22]. Another group of solution concepts, based on the "bargaining theory" initiated by Aumann and Maschler [4], is of particular interest and includes a variety of bargaining sets, [4] (with respect to the bargaining set $M_1^{(1)}$),

see in particular Davis and Maschler [8], and Peleg [27]), the kernel, Davis and Maschler [7], and the nucleolus, Schmeidler [34].

The bargaining sets and the kernel, as solution concepts, are of particular interest because Pareto optimality, which seems to be a necessary characteristic for stability in solution concepts such as the core, is no longer required in the bargaining sets and the kernel. This represents a radical departure from the basic idea that players in a cooperative game will try to make the most out of it.

1.2 Applications to Economics

The representation of a general equilibrium model of an economy by a cooperative game can be considered as a significant conceptual breakthrough (e.g., [35, pp. 505-506]) because a variety of general equilibrium problems can be studied in a context free of any specific reference to resource allocation mechanisms or institutional arrangements. The idea is that, once the problem of establishing a conceptual equivalence between a model of an economy and a cooperative game has been solved, solutions to that game could guide us on what stable institutional arrangements are compatible with the given economy in the sense that stability of such institutional arrangements will depend only on the characteristics of that economy (see [25], Section 4.7) such as, initial distribution of resources, technologies, and preferences.^(*) To be more precise, given the set of stability criteria used to define a particular solution concept, what allocations meet such criteria will be determined

(*) This, of course, does not mean that such institutional arrangements will, necessarily, be unique for a given economy and a given solution concept.

by the characteristics of the economy. But then, any institutional arrangement that leads to allocations which satisfy such criteria can be considered stable, in that sense, and such stability of institutional arrangements will depend only on the characteristics of the economy in the same way that the relevant solution concept does.

To demonstrate this, let us consider a "symmetric market game" (so called by Shapley [36] for reasons that will become apparent from its description).

There are two "types" of players such that players of opposite types are complementary (i.e., their cooperation can be beneficial), while players of the same type are not.

A game like this has been used to represent (a) the market of some commodity [25, Sections 64.1, 64.2], and [36], with players of one type representing the sellers while players of the other type represent the buyers of the commodity;^(*) or (b) an Edgeworth pure exchange economy [40].

The compatibility of institutional arrangements such as monopoly, oligopoly, monopsony, or perfect competition, with an economic environment representable by a symmetric market game will become apparent if one considers a solution to the latter, such as the core, which will depend only on the number of players of each type, their endowments of resources, and their preferences, all descriptive characteristics of the economic environment. The only stable institutional arrangements in this case, will be those which yield utility vectors corresponding to the core of the respective symmetric market game.

^(*) That utility is transferable is essential for this representation.

Apparently, and in applying cooperative game theory to economic analysis, one must be concerned, like with any other situation that involves conflict of interest, with the two fundamental questions of game theory, that is: (a) what cooperative game to use in the analysis, and (b) what solution concepts to apply.

Economies as Cooperative Games

The example of the symmetric market game, presented above, used to represent the market of a commodity, relies upon the assumption that utility is transferable. This is an assumption used quite often in the literature because in many cases it simplifies considerably the analysis. For that matter, to represent the market of a single commodity, which is only a partial equilibrium model, as a cooperative game, the use of an assumption like this, or a similar one, may be necessary. Although the concept of a transferable utility used to represent certain situations of conflict of interest by a cooperative game may not result in any loss of generality, as it has been pointed out by Scarf [33], and then by Debreu and Scarf, this concept "has remained foreign to the mainstream of economic thought" [9, p. 236].

In [37], Shapley and Shubik showed that transferability of utility is not essential for the Von Neumann and Morgenstern theory of games, and that once the assumption that utility is transferable is dropped, side payments can be made in any commodity. Then, following Luce and Raiffa [19, Section 10.4], Aumann and Peleg [5], and Aumann [1], Scarf [33] and subsequently Debreu and Scarf [9] considered models of economies as cooperative games in characteristic function form without side payments and without transferable utility (for a survey on cooperative games without side payments see Aumann [3]). From there on, it has been

taken, almost, for granted that such cooperative games could adequately represent the entire range of conflicts of interest that may exist in a given economic environment. However, and as we have already pointed out above, it was shown by Rosenthal [32] that this is not necessarily the case. What is particularly interesting about the examples used in [32] is that they are examples which involve some kind of externalities.

Although cooperative games in characteristic function form without transferable utility and without side payments fit well into "the mainstream of economic thought," the fact remains that certain characteristics of the Von Neumann and Morgenstern theory of games, which may be considered quite restrictive, have been carried over to the new theory. In particular, and given that economic agents carry on a variety of economic activities, and they do not do so, necessarily, with the same group of people, one may well raise the question whether, this new theory, in trying to stay close to the classical theory of games, has not ignored a variety of conflicts of interest that may exist when people decide that it could be to their own best interest to carry on different economic activities as members of different groups. This, of course, involves a number of game theoretic questions, and in particular questions concerning the cooperative game that could represent such conflicts of interest.

Core Theory

Because of its relationship to the set of competitive equilibria of an economy, the core, as a solution concept to a cooperative game that could represent conflicts of interest in that economy, has been the major solution concept utilized in applications of cooperative game theory to economics.

The connection between the core of a "symmetric market game" used to represent an Edgeworth pure exchange economy, and Edgeworth's contract curve in [10], was made, first, by Shubik [40].

In [33], H. Scarf, and subsequently Debreu and Scarf [9], defined the core, with respect to a general pure exchange economy with a large, but finite, number of economic agents and many commodities, without using the assumption of transferable utility (as we have mentioned above), and proved "Edgeworth's proposition" that, in the limit, the core will shrink to the set of competitive equilibria of an economy as the number of economic agents is, appropriately, enlarged.

In [2], Aumann introduced the concept of an "atomless economy," that is, an economy with a continuum of traders, as a way to deal with the problem that under perfect competition the influence of each individual economic agent should be negligible. He then proved that the core, with respect to an "atomless economy" will be identical to the set of competitive equilibria of that economy.

Similar results to the above have been obtained in the literature when production is introduced into the model of an economy. The reader is referred to Hildenbrand [16] and some of the references therein.

Although the core, defined with respect to an economy without public goods and externalities has some interesting properties relative to the set of competitive equilibria of such economies, this may no longer be the case when public goods and externalities enter into the analysis.

The core with respect to an economy with pure public goods, was defined and shown to contain the set of Lindahl equilibria of that economy by Foley [12], and [13]. In [24], Muench has shown that the core, defined with respect to such economies, does not necessarily

converge to the set of Lindahl equilibria of those economies and, in fact, it may be very large for an economy with a continuum of economic agents.

In [39], Shapley and Shubik have shown that the core defined with respect to an economy with externalities may be empty. In fact, Ellikson [11] has showed that this result may hold when public goods are crowded, and that, in such cases, even if the core is not empty it is not necessary that it will contain the Lindahl equilibria of the respective economy.

These last results have prompted a series of articles dealing with questions concerning the core, and alternative definitions for it, for economies with public goods and externalities. For the record we should mention the works of M. Pauly [26], D. Roberts [29], and [30], D. Richter [28], Champsaur, Roberts, and Rosenthal [6], R. Rosenthal [31], D. Starret [42], and a survey article by Milleron [23].

1.3 A Brief Statement of the Problem

It should be apparent by now, from the discussion in the preceding two sections, that, in dealing with cooperative game theory and its applications to economics, one faces a twofold problem.

Cooperative games in characteristic function form, in general, may be inadequate to capture, and deal with, certain situations of conflict of interest. In particular, this may be the case for an economy in the presence of (a) externalities and public goods, and (b) a multiplicity of economic activities which require the interaction of economic agents from groups with opposing interests.

This raises the question whether it is possible to define cooperative games in a form where the above inadequacies will be removed. Apparently, and given that such games can be defined, still, there will be a question about solution concepts to such games.

Independently of the preceding aspect of the problem, and with respect to solution concepts of cooperative games, the core is easy to define and use, and its stability properties are quite plausible. However, and because many games have an empty core, there is a need for alternative solution concepts. This raises the question of what set of stability criteria should be used for such solution concepts.

Our purpose in this work is to provide some answers to these questions.

1.4 Mathematical Notation

Throughout this work, and unless otherwise specified, the following notation is used exclusively with the meaning given here.

Elementary Symbols

- $\Rightarrow$ implication connective (reads: "implies")
- $\Leftrightarrow$ equivalence connective (reads: "if and only if")
- $\exists$ existential quantifier (reads: "there exists")
- $\forall$ universal quantifier (reads: "for all")
- $\equiv$ identity symbol (reads: "is identical to" or "by definition is equal to." Used, in general, when some notation is introduced for the first time.)
- $\in$ symbol for set membership (reads: "belongs to")
- $\notin$ symbol for denial of set membership (reads: "does not belong to")

$\exists$:	"such that"
$\subset$	"is a proper subset of"
$\subseteq$	"is a subset of"
$\cap$	symbol for set intersection
$\cup$	symbol for set union
$ a $	$\equiv \text{Max}\{a, -a\}$, absolute value of any real number a .

Sets

$\emptyset$	empty set
E^n	n -dimensional Euclidean space where n is a positive integer
E_+^1	the set of non-negative real numbers
E_+^n	n -fold Cartesian product of the set E_+^1 , where n is a positive integer
$\{1, \dots, n\}$	the finite set consisting of all positive integers 1 through n .
$X^{(n)}$	n -fold Cartesian product of any set X which has not been specified to be the set E^1 , or E_+^1 .
$X^1 \times \dots \times X^n$	Cartesian product of the sets X^i , $i \in N$, where $N \equiv \{1, \dots, n\}$; we often write $\prod_{i \in N} X^i$ to denote the Cartesian product of such collection of sets.
$(A-B) \equiv \{x \in A: x \notin B\}$,	set difference between any two sets A and B .

Vectors

$(x^1, \dots, x^n)$	explicit description of any vector $x \in E^n$
x^i	the i -th element of a vector $x \in E^n$
$p^i(x)$	the projection into the i -th coordinate of any vector $x \in E^n$, i.e., $p^i: E^n \rightarrow E^1$, and $p^i(x) \equiv x^i$, the i -th element of x .

$(A^i)_{i \in c}$ an ordering of the objects A^i , where, for a set N , $N \neq \emptyset$, $c \subseteq N$, and for each $i \in N$ there corresponds an object A^i ; in particular, if x is a vector, say $x \in E^n$, then $(x^i)_{i \in c}$ represents the subvector of elements in x corresponding to any subset $c \subseteq \{1, \dots, n\}$.

$d(x, y)$ the Euclidean distance between any two vectors $x, y \in E^n$;

$d(x, A) \equiv \text{Infimum } \{d(x, y) : y \in A\}$, for any $x \in E^n$, and any A ,

$A \subseteq E^n$, $A \neq \emptyset$.

For any two vectors $x, y \in E^n$, we write

$(x > y) \iff (x^i > y^i, \forall i \in \{1, \dots, n\})$,

$(x \geq y) \iff (x^i \geq y^i, \forall i \in \{1, \dots, n\})$, and

$(x \leq y) \iff ((x \geq y) \text{ and } (x \neq y))$.

CHAPTER TWO
ECONOMIC ENVIRONMENTS WITH EXTERNALITIES
AND PUBLIC GOODS AS COOPERATIVE GAMES

2.1 Introduction

In viewing models of economies as cooperative games, the most common approach in the literature is to assume that individual groups of economic agents pool together their (productive) resources and carry on, collectively, every economic activity (i.e., trades, production, and distributions) that they decide to undertake, without any interaction with economic agents from other groups. Thus, the cooperative games that could emerge from this assumption have a "built-in" restriction. Players, in such games, are not allowed to use strategies which imply that they can carry on, simultaneously, different economic activities as members of different groups. One, then, may raise the question whether we can view models of economies, in particular, economies with a variety of economic activities, as cooperative games that do not have the above "built-in" restriction. The task of answering this question, which requires answers to a series of other game theoretic questions, is undertaken in this chapter. In particular, our concern here is with a class of economic environments that could serve as general models for analyzing externality phenomena of any kind, including the production of public goods as a special case of economic activities which are the source of externalities.

Apparently, if we were to start out with some particular cooperative game, and if, by restricting the strategies which the players, in such games, are allowed to use, we could obtain a new, simpler game, that has identical solution concepts (in both content and size) as the original one, then, and as far as our interest lies with those solution concepts, this new, simpler game will be preferable for use in our analysis. However, solution concepts, such as the core, are not necessarily invariant between any two cooperative games that differ from each other only on the set of strategies allowed to the players in each game. Therefore, to decide between any two games, as above, implicitly, may involve a decision on the content or the size of some solution concept to such games.

To carry this argument over to the way that we view models of economies as cooperative games, an assumption which, in effect, limits the strategies that the set of individuals in an economy are allowed to use, may result in a cooperative game whose solution concepts may be different (in content or size) from the cooperative game which could result without such restrictions on strategies. In turn, this may affect some of the results obtained in the literature of the "theory of the core." This is the first of the two major reasons that we want to consider the possibility that a model of an economy could be viewed as a cooperative game where individual economic agents are allowed to use strategies which imply that these agents can carry on, simultaneously, different economic activities as members of different groups.

The second major reason that we are interested in viewing models of economies as cooperative games with the above characteristic, is that, in the first place, such games could capture existing conflicts of

interest that games with the "built-in" restriction, mentioned above, cannot. Thus, the first important step towards this direction will be to model economic environments in a way that such conflicts of interest will be clearly identifiable. This step is undertaken in Section 2.2, below.

To get an idea of what these conflicts of interest might be, let us consider the following example. There are three individuals, which we shall call Mr. A, Mr. B, and Mr. C. Mr. A owns a piece of land where he can grow some product, say potatoes. Mr. B owns a tractor, and Mr. C owns a truck. The situation is the following. Mr. A and Mr. B can put their resources together (say land which is not used at the present by Mr. A, and Mr. B's tractor) to increase the production of potatoes. However, neither Mr. A nor Mr. B have any use for a large amount of the extra potatoes that they can produce, and these potatoes cannot be sold at the local market. Now, if Mr. C agrees to use his truck, any amount of potatoes, so produced, can be transported and sold at a different location. The problem is, that Mr. B and Mr. C are not in very good terms with each other, and neither one of them is willing to enter into an agreement that will involve the other. In terms of preference, the various alternatives that could emerge from this situation are rated as follows.

Mr. A will be better off if he cooperates with Mr. B, provided that he ends up with some extra money, i.e., provided that some of the extra potatoes are transported and sold, and he receives part of the money. Otherwise, he is indifferent between cooperating with Mr. B. or not.

Mr. B rates the various alternatives exactly like Mr. A, with one provision. He (Mr. B) will not have to enter into any agreements with Mr. C. Any agreement that involves the participation of both, Mr. B and Mr. C, no matter what its monetary outcome, will make Mr. B worse off than what he will be without entering into any agreements.

Mr. C will be better off if (a) he enters into an agreement with Mr. A for the use of his truck, and (b) after discounting for costs, he could end up with some extra money (which is always possible) provided that such agreement does not involve Mr. B. Any agreement that involves the participation of both, Mr. B and Mr. C, no matter what its monetary outcome, will make Mr. C worse off than what he will be without entering into any agreements.

Finally, let us suppose that any arrangements, with third parties, that have not been, explicitly, introduced in the above example are not possible.

There are two major economic activities involved in this example (actually, and counting "holding to one's own resources," as a major economic activity, there are three) namely (a) production of potatoes, and (b) transportation and marketing of potatoes.

To attain a Pareto optimal allocation, in this example, we must have that Mr. A and Mr. B carry on, collectively, the activity of producing potatoes, while Mr. A and Mr. C carry on, collectively, the activity of transporting and marketing potatoes. Any other arrangements will not yield a Pareto optimal allocation. In particular, the grand coalition (consisting of all three individuals) will produce an infinite loss of utility to both Mr. B and Mr. C.

What this example indicates is the following. Without a provision that allows individual economic agents, acting as players in a cooperative game, to use strategies which imply that they can carry on, simultaneously, different activities as members of different coalitions, the cooperative games, that could emerge from a model of an economy, might not capture the entire range of conflict of interest.

Apparently, and although from the point of view of economic theory it is not difficult to identify the process that could lead to Pareto optimal allocations in the above example (we have already done so, above); from a game theoretic point of view this, it is not very easy. In particular, it is not so clear what the strategies of the various coalitions will be in an example like the above. The preliminary work for identifying such strategies, in a general economic model, is presented in Sections 2.3, and 2.4, below. The formal presentation of the cooperative games that could emerge from economic environments with conflicts of interest that could include situations like the one in the above example, is made in Section 2.5, below, while, the problem of defining strategies for coalitions in such games is dealt with, in Section 2.6.

Because a comparison between a cooperative game with the "built-in" restriction, mentioned at the beginning of this introduction, with a cooperative game without this restriction, is quite interesting, and because subsequently in this work we are going to use both types of cooperative games, with respect to a given economic environment, we present the former of the two cooperative games, in Section 2.7, below.

Finally, and because the class of economic environments utilized in this work share some common characteristics with the class of "coalition production economies" introduced by Hildenbrand [14], [15], and [16], we compare the two, briefly, in Section 2.8.

2.2 Economic Environments

Our concern here is with the description of a class of economic environments (economies) which could serve as general models for analyzing such aspects of economic activity as (a) the interdependencies of the actions of economic agents, (b) the various costs associated with such actions, and (c) externality phenomena of any kind, including the production of public goods, as a special case of an externality.

The notion of an economic environment introduced below, although a much more basic concept than similar notions utilized in the literature, could be described quite accurately through the following definition used by Ledyard [18, p. 1608]. Furthermore, some of the terminology used in [18] is also utilized in this work. Briefly speaking then, an economic environment "is a collection of agents, a description of their feasible actions, the outcomes which can result from these actions, and the agents' preferences for these outcomes."

The collection of economic agents, that we deal with, consists of a finite set of individuals N , indexed by $i = 1, \dots, n$; $n \geq 2$. Subsets of the set N , denoted by the letter " c ," are called coalitions, and the set of all subsets of N will be denoted by C_N . Thus, C_N represents the set of all coalitions possible in the economy. A partition of the set N into non-empty coalitions is called a coalition structure. A coalition structure is denoted by the Greek letter τ , and the set of all coalition structures, possible, by T_N .

There are finitely many commodities, indexed by $s = 1, \dots, m$; $m \geq 2$. Thus, the commodity space, denoted by Q , is a subset of the m -dimensional Euclidean space E^m , i.e., $Q \subseteq E^m$. We distinguish

commodities into different "types," such as "pure private goods," "pure public goods," and so on. Hence, the set $\{1, \dots, m\}$ could be partitioned into subsets, say $\{m_1, \dots, m_r\}$, where each subset m_j , $j = 1, \dots, r$, will represent all commodities of the same "type." In turn, through such partition, we can divide the commodity space into different subspaces, each corresponding to a different type of commodity. For the class of economic environments considered here, and for the purposes of this work, it suffices to say that Q_p represents the subspace of Q corresponding to all "pure private goods" in the economy. (*)

Each individual $i \in N$ is endowed with a vector of resources equal to ω^i (his initial endowment) such that $\omega^i \in Q_p$. In other words, we deal with a private ownership economy where each individual's vector of initial endowments consists only of pure private goods. The vector $(\omega^i)_{i \in N}$ then represents the initial allocation in the economy.

Each individual $i \in N$ plays two roles in the economy: (a) the role of a consumer, and (b) he participates in the production process of the economy.

As a participant in the production process of the economy, an individual could, individually or collectively with others, undertake a

(*) Given that, at some period of time, a total amount of $\bar{x}_s$ units of commodity s is available in the economy, we say that commodity s is a pure private good if, whenever an individual $i \in N$ uses x_s^i units of commodity s , for some purpose, then the total amount that will be left in the economy, for any purpose, is equal to $\bar{x}_s - x_s^i$.

number of "activities," indexed by $j = 1, \dots, k$. Let J denote the set of all possible activities in the economy. Our understanding here is that any process which, if followed by some non-empty set of individuals in the economy, could, in some way, change the initial allocation $(w^i)_{i \in N}$, should constitute an activity that must be included in the set J . In particular then, the production of any commodity, trades among individuals, voluntary transfer of resources to other individuals, disposal of resources, even holding to one's own resources, etc., are processes which could change the initial allocation $(w^i)_{i \in N}$. Hence, these are processes which must be included in the set J if one is to proceed under the assumption that such processes would take place in the economy.

An individual who undertakes an activity $j \in J$ is involved in a twofold decision process.

1. He decides what part of his endowment of resources he will devote to that activity.
2. He decides on the coalition as a member of which he is going to undertake that activity.

Let ℓ_j^i denote the input of resources that individual i devotes to activity j and let c_j^i denote the coalition as a member of which individual i has decided to undertake activity j . Then, the pair $(\ell_j^i; c_j^i)$, $\ell_j^i \in Q_p$, $c_j^i \in C_N$, $i \in c_j^i$, represents the twofold decision process by individual i with respect to activity j , and it is called an action by individual i with respect to activity j . An action $(\ell_j^i; c_j^i)$, $i \in N$, $j \in J$, will be denoted by $\underline{a}_j^i$, i.e., $\underline{a}_j^i \equiv (\ell_j^i; c_j^i)$. A list of actions, one for each activity $j \in J$, by any individual $i \in N$ constitutes an individual action and it is denoted by $\underline{a}^i$, i.e., $\underline{a}^i \equiv (\underline{a}_j^i)_{j \in J}$.

That an individual $i \in N$ must decide on an action for every activity $j \in J$ does not, necessarily, mean that he actually undertakes every single activity in the set J and carries it at a meaningful level. For example, an individual may choose just himself for the coalition that he is going to undertake an activity and the devotion of no resources for that activity. This means that the pair $(0; \{i\})$ is a possible action for any individual $i \in N$ with respect to any activity $j \in J$. From now on, we will proceed under the assumption that if $\underline{a}_j^i = (0; \{i\})$, for any $i \in N$, and any $j \in J$, then individual i does nothing with respect to activity j .

A list of individual actions, one for each individual $i \in N$, constitutes a joint action in the economy, and it is denoted by $\underline{a}$, i.e., $\underline{a} \equiv (\underline{a}_j^i)_{i \in N}$.

Feasibility of individual actions in the economy requires that the total amount of resources devoted to each and every activity $j \in J$ by an individual be exactly equal to that individual's vector of initial endowments. This notion of individual feasibility follows from the fact that "disposal of commodities" and "holding to one's own resources" could constitute activities which must be included in the set J if, in our analysis, the inclusion of such activities becomes necessary.

The general rule for feasibility of joint actions in the economy is that the total amount of resources devoted to each and every activity $j \in J$ by each and every individual $i \in N$ must be equal to the total amount of resources available, initially, in the economy. However, due to our assumptions about the set of activities J , a joint action will not be feasible if it does not consist of feasible individual actions. For example, a feasible joint action which does not consist of feasible

individual actions will imply that some transfers (voluntary or not) have to take place. But this implies that there exists some activity in the set J that accomplishes those transfers. Therefore, feasibility of joint actions, in the class of economies that we consider here, coincides with feasibility of the individual actions that comprise each of them.

The technology of the economy is described through an aggregate production correspondence which transforms each joint action into a set of final outcomes. Final outcomes are in the form of allocations. Therefore, if $\psi(\cdot)$ denotes the aggregate production correspondence of the economy then ψ maps joint actions into the n -fold Cartesian product of the commodity space.

There are several observations that we should make relative to the above specification of a technology.

1. As we have already pointed out, production of any commodity, if such is to take place in the economy, constitutes an activity that must be included in the set J . However, and because economic agents in the set N are individuals, "producers" or "firms," if such are to be present in the economy, will consist of coalitions formed by a set of individuals for the purpose of undertaking the relevant activities that produce certain commodities. In other words, only what we may call coalition firms could be present in the economy. Furthermore, such coalition firms are not entities which could exist independently of each individual's twofold decision process described earlier in this section.

2. Because our concern here is with economic environments that could serve as general models for analyzing the behavior of economic

agents if externalities in production are present, it may not be possible to assign separate production technologies either to individual coalitions (which is the case for coalition production economies) or to each individual activity in the set J (which is the case for an Arrow-Debreu economy). For this reason we only require that an aggregate technology be specified for the economy.

3. For the aggregate production correspondence $\psi(\cdot)$ to yield allocations, the set of activities J must contain (as we have already pointed out) specific activities that will make possible the transformation of any joint action into a set of allocations. Apparently, such activities produce something, namely allocations. Hence, our use of the term "production" to characterize the correspondence $\psi(\cdot)$ is justified if we were to interpret "production" in a wide sense.

4. That the relationship $\psi(\cdot)$ is best characterized as a correspondence, and not as a (single valued) function, will follow from the fact that even if a coalition distributes whatever it produces among its members, many distributions of such products may be possible. However, if only one distribution is possible for each joint action in the economy then $\psi(\cdot)$ will be equivalent to an "aggregate" production function.

We come now to the role of each individual in the economy as a consumer. Because our concern here is that externalities in consumption, if such are present in the economy, should be taken into account, we specify that individual preferences are defined over consumption sets that satisfy two conditions.

1. Each individual's consumption set consists of a subset of the n -fold Cartesian product of the commodity space, where n is the number of individuals in the economy.

2. Each individual's consumption set contains all final allocations that could result from feasible joint actions.

Apparently, the definition of individual preferences over consumption sets that satisfy the above two conditions does not, necessarily, mean that such preferences cannot be "selfish or individualistic," (in the sense of Hurwicz [17]). It just indicates that if externalities in consumption are present in the economy then we can take into account such externalities.

For future reference we consolidate the description of an economic environment given above and the related concepts in the following notation.

Notation

- $N \equiv \{1, \dots, n\}$, the set of economic agents; $n \geq 2$.
- $C_N \equiv \{c: c \subseteq N\}$, the set of coalitions, given N .
- $(N-c)$ the complement, in N , of any coalition $c \in C_N$.
- T_N the set of coalition structures, given N , (the set of partitions of N). Elements of the set T_N are denoted by the Greek letter τ .
- $C_N^i \equiv \{c \in C_N: i \in c\}$, the set of coalitions that contain individual i as a member, $i \in N$.
- Q the commodity space; $Q \subseteq E^m$; $2 \leq m < \infty$
- Q_p the subspace of Q concerning all private goods; $Q_p \subseteq E^{m'}$, i.e., there are m' private goods, with $1 \leq m' \leq m$.
- $Q^{(n)}$ the n -fold Cartesian product of Q .
- ω^i the i -th agent's vector of initial endowments; $\omega^i \in Q_p$;
 $\omega^i \in E_+^{m'}$

$\omega \equiv (\omega^i)_{i \in N}$, the initial allocation of resources.

$J \equiv \{1, \dots, k\}$ the set of activities.

A_j^i the i-th agent's set of actions with respect to the j-th activity, $i \in N$, $j \in J$; $A_j^i \subseteq (Q \times C_N^i)$.

We use the notation $\underline{a}_j^i$ to denote an element of the set A_j^i , and we call such element an action by the i-th agent with respect to the j-th activity. We write $\underline{a}_j^i = (\ell_j^i; c_j^i)$ in order to indicate that ℓ_j^i represents the input of resources by the i-th agent into the j-th activity and that c_j^i represents the coalition component of $\underline{a}_j^i$. It is assumed that $(0; \{i\}) \in A_j^i$,

$\forall i \in N$, and $\forall j \in J$.

$A^i \equiv \Pi_{j \in J} A_j^i$, the i-th agent's set of individual actions. Elements of the set A^i are denoted by $\underline{a}^i$, i.e., $\underline{a}^i \equiv (\underline{a}_j^i)_{j \in J}$.

$A \equiv \Pi_{i \in N} A^i$, the set of joint actions. Elements of the set A are denoted by $\underline{a}$, i.e., $\underline{a} \equiv (\underline{a}^i)_{i \in N}$.

$\hat{A}^i \equiv \{a^i \in A^i: \underline{a}_j^i = (\underline{a}_j^i)_{j \in J}, \underline{a}_j^i = (\ell_j^i; c_j^i), 0 \leq \ell_j^i \leq \omega^i, \forall j \in J, \text{ and } \sum_{j \in J} \ell_j^i = \omega^i\}$, the i-th agent's set of feasible individual actions.

$\hat{A} \equiv \Pi_{i \in N} \hat{A}^i$, the set of feasible joint actions.

$\psi: A \rightarrow Q^{(n)}$, the aggregate production correspondence.

$Y \equiv \bigcup_{\underline{a} \in A} \psi(\underline{a})$, the set of final outcomes (allocations); elements of the set Y are denoted by y .

$\hat{Y} \equiv \bigcup_{\underline{a} \in \hat{A}} \psi(\underline{a})$, the set of feasible final outcomes (allocations).

$\bar{Y}^i$ the i-th agent's consumption set; $\bar{Y}^i \subseteq Q^{(n)}$, $\hat{Y} \subseteq \bar{Y}^i$, $i \in N$.

$\bar{Y} \equiv (\bar{Y}^i)_{i \in N}$.

$\succeq_i$ the i -th agent's preference ordering, $i \in N$; $\succeq_i \subseteq \bar{Y}^i \times \bar{Y}^i$;
 for $\bar{y} \in \bar{Y}^i$, and $y \in \bar{Y}^i$, we write $\bar{y} \succeq_i y$ to mean that
 $\bar{y} \succeq_i y$ but not $y \succeq_i \bar{y}$; $\bar{y} \sim_i y$ means that $\bar{y} \succeq_i y$ and
 $y \succeq_i \bar{y}$.

$R \equiv (\succeq_i)_{i \in N}$, a profile.

$\underline{e} \equiv (N, \omega; Q; J, A, \psi; \bar{Y}, R)$, an economic environment (economy).

According to the notation introduced above, an economic environment $\underline{e}$ will be completely specified if we specify an 8-tuple
 $(N, \omega; Q; J, A, \psi; \bar{Y}, R)$.

In concluding this section, let us recall that the set of activities J , in a given economic environment $\underline{e}$, must include specific activities that yield final allocations. Furthermore, such activities do not have to be entirely costless. Therefore, an economic environment $\underline{e}$ could serve as a model for situations where the attainment of a final distribution, through such activities as trades, marketing, and so on, could be costly. Furthermore, and given that it is possible for any single activity to be carried on by coalitions of various sizes, then, "costs to size," if such are present, will be shown by their influence on the set of final outcomes, as we vary joint actions only with respect to the size of various coalitions. Therefore, an economic environment $\underline{e}$ could also be used as a model for studying such costs.

2.3 Consistent Joint Actions

Given an economic environment $\underline{e} = (N, \omega; Q; J, A, \psi; \bar{Y}, R)$, the set of joint actions A contains a number of elements which could be considered "consistent" in terms of the way that the set of individuals N select the coalition components of their action with respect to the

various activities in the set J . In this section we examine such joint actions.

Using the notation introduced in the preceding section, let us write an arbitrary joint action $\underline{a}$, $\underline{a} \in A$, in terms of the actions $\underline{a}_j^i$, $j \in J$ that comprise each individual action $\underline{a}^i$, $i \in N$, which, in turn, comprise the joint action $\underline{a}$, i.e., let us write any $\underline{a} \in A$, as $\underline{a} = ((\underline{a}_j^i)_{j \in J})_{i \in N}$. Now, given a joint action $\underline{a} \in A$, the actions $\underline{a}_j^i$ consist of two components, i.e., $\underline{a}_j^i = (\ell_j^i; c_j^i)$, for each $i \in N$ and for each $j \in J$. Furthermore, for each action $\underline{a}_j^i$, $i \in N$, $j \in J$, and relative to the coalition component c_j^i of such action, there are two, mutually exclusive, possibilities in any given joint action $\underline{a}$.

1. There exists an individual s , $s \in c_j^i$, $s \neq i$, whose action $\underline{a}_j^s, \underline{a}_j^s = (\ell_j^s; c_j^s)$, is such that $c_j^s \neq c_j^i$.
2. For each individual s , $s \in c_j^i$, the action $\underline{a}_j^s, \underline{a}_j^s = (\ell_j^s; c_j^s)$, is such that $c_j^s = c_j^i = c$.

In words, and in the sense that a joint action $\underline{a} \in A$ consists of the actions selected by each individual $i \in N$ with respect to each activity $j \in J$, we can say the following about the above two cases. In the first case, the coalition component c_j^i , selected by individual i , in his action with respect to activity j , contains a member who, in his action with respect to the same activity j , has selected a different coalition component. In this sense then, the actions $\underline{a}_j^i$ and $\underline{a}_j^s$ in the joint action $\underline{a}$ are incompatible with each other. In the second case, the coalition component c_j^i selected by individual i in his action with respect to activity j coincides with the coalition component c_j^s selected by each member s , of coalition c_j^i , in his action with respect

to the same activity j . Hence, the actions $\underline{a}_j^i$ and $\underline{a}_j^s$, for each $s \in c_j^i$, are compatible with each other in the joint action $\underline{a}$.

From now on, and whenever, for a given joint action $\underline{a} \in A$, there exists a non-empty coalition c , $c \in C_N$, and there exists an activity j , $j \in J$, such that for each $i \in c$, the action $\underline{a}_j^i$ satisfies the second case above (i.e., whenever the actions $\underline{a}_j^i$ and $\underline{a}_j^s$ in $\underline{a}$ are compatible with each other for each pair of individuals in c) we shall say that the joint action $\underline{a}$ admits coalition c with respect to activity j , or that c is an $\underline{a}$ - admissible coalition. In other words, a non-empty coalition $c \in C_N$, is an $\underline{a}$ - admissible coalition, for some $\underline{a} \in A$, if $\underline{a}_j^i = (\ell_j^i; c)$, $\forall i \in c$, some $j \in J$.

Let us observe that whether or not a joint action $\underline{a} \in A$ admits any non-empty coalitions, is not related to the particular process that the set of individuals N , is an economy $\underline{e}$, follow in selecting their actions with respect to the various activities in the set J . Hence, admissibility, of certain coalitions by certain joint actions in the set A , as defined above, should be seen as a characteristic of joint actions, inherent in the definition of an economic environment $\underline{e}$.

Given a joint action $\underline{a} \in A$, and using the above notion of admissibility of coalitions by joint actions, we can say that, in the first of the two cases which are possible, with respect to an action $\underline{a}_j^i = (\ell_j^i; c_j^i)$, discussed above, the coalition c_j^i is not an $\underline{a}$ - admissible coalition. However, to say that a joint action $\underline{a} \in A$ contains an action $\underline{a}_j^i = (\ell_j^i; c_j^i)$ such that c_j^i is not an $\underline{a}$ - admissible coalition means that there is some inconsistency with the joint action $\underline{a}$. In other words, at least one individual $i \in N$ will not be able to carry on activity j in the way that he has selected to do so. Therefore, of

interest are joint actions in the set A where such inconsistencies are absent. From now on, we will refer to joint actions $\underline{a} \in A$ where the coalition component c_j^i , of each action $\underline{a}_j^i$, $\underline{a}_j^i = (\ell_j^i; c_j^i)$, $i \in N$, $j \in J$, is an a - admissible coalition, as consistent joint actions.

For future reference, we incorporate the concepts introduced in this section, and the related notation, in the following definition.

Definition 2.1: For a given economic environment $\underline{e} = (N, \omega; Q: J, A, \psi; \bar{Y}, R)$:

- (a) A joint action $\underline{a}$, $\underline{a} \in A$, admits coalition c , $c \in C_N$, with respect to activity j , $j \in J$, i.e., c is an a - admissible coalition, if and only if: $\underline{a}_j^i = (\ell_j^i; c)$, $\forall i \in c$.
- (b) $C(\underline{a}) \equiv \{c \in C_N: \exists j, j \in J, \text{ and } \underline{a}_j^i = (\ell_j^i; c), \forall i \in c\}$, the set of a - admissible coalitions, for each $\underline{a} \in A$.
- (c) A joint action $\underline{a}$, $\underline{a} \in A$, is a consistent joint action if and only if $\underline{a}_j^i$, $\underline{a}_j^i = (\ell_j^i; c_j^i)$, is such that $c_j^i \in C(\underline{a})$, $\forall i \in N$, and $\forall j \in J$.
- (d) $A^* \equiv \{\underline{a} \in A: \underline{a} \text{ is a consistent joint action}\}$, the set of consistent joint actions for the economy $\underline{e}$.
- (e) $\hat{A}^* \equiv \hat{A} \cap A^*$, the set of feasible consistent joint actions for the economy $\underline{e}$.
- (f) $\hat{Y}^* \equiv \bigcup_{\underline{a} \in \hat{A}^*} \psi(\underline{a})$, the set of final outcomes (allocations) resulting from feasible consistent joint actions in the economy $\underline{e}$.

It is of interest here to observe the following. In an economic environment $\underline{e}$, each individual $i \in N$, cannot undertake more than a single action with respect to each activity $j \in J$. On the other hand, in a consistent joint action $\underline{a} \in A^*$ the coalition component c_j^i of each action $\underline{a}_j^i$, $i \in N$, $j \in J$, is an $\underline{a}$ - admissible coalition (part (c) of the above definition). Therefore, to say that a joint action $\underline{a} \in A$ is a consistent joint action is equivalent to saying the following. The set of individuals N , form a coalition structure τ_j with respect to each activity $j \in J$. Apparently, the coalition structure τ_j , as above, does not have to be the same for every activity $j \in J$, in a given joint action $\underline{a} \in A^*$. It will follow then that, for a consistent joint action $\underline{a} \in A^*$ the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$ consists precisely of the union of all coalitions in each and every coalition structure which is formed with respect to each activity $j \in J$. For future reference, we state the above properties of consistent joint actions as a Lemma whose proof follows directly from the definition of an economic environment $\underline{e}$ in conjunction with Definition 2.1, above.

Lemma 2.1: For a given economic environment $\underline{e}$:

- (a) $(\underline{a} \in A^*) \iff (\text{for each } j, j \in J, \exists \tau_j, \tau_j \in T_N,$
 $\exists: \underline{a}_j^i = (\ell_j^i; c) \text{ for each } c \in \tau_j \text{ and for each } i \in c);$
- (b) $(\underline{a} \in A^*, \text{ and } \tau_j, j = 1, \dots, k, \text{ are the corresponding to each}$
 $\text{activity coalition structures}) \implies (C(\underline{a}) = \bigcup_{j \in J} \tau_j).$

To get a better understanding about consistent joint actions in an economy $\underline{e}$, let us suppose that the set of individuals N , in selecting their actions with respect to each activity $j \in J$, follow a process where each action $\underline{a}_j^i = (\ell_j^i; c_j^i)$, $i \in N$, $j \in J$, could be interpreted as

a proposal. In particular, a proposal by individual i to the other individuals in the set c_j^i to the effect that they should carry on, collectively, activity j . In this sense, a joint action $a \in A$ will consist of $n \times k$ proposals (i.e., n , the number of individuals in the set N , times k , the number of activities in the set J). Hence, a non-empty coalition $c \in C_N$ will be an a - admissible coalition if, with respect to some activity $j \in J$, there is a coincidence of proposals among the individuals in c . Therefore, to say that a joint action $a \in A$ is a consistent joint action will mean that such coincidence of proposals is universal, i.e., the proposal of each individual $i \in N$ with respect to each activity $j \in J$, to the members of coalition c_j^i coincides with the respective proposals of such members. Of particular interest then, are processes which, in the sense described above, will exclude the possibility that the set of individuals N would end up undertaking a joint action where there is not universal coincidence of proposals. In other words, processes where the set of individuals N could be undertaking only consistent joint actions. As we shall see in Sections 2.5 and 2.7, below, the bargaining process that takes place in a cooperative game could be interpreted as such process. Hence, cooperative games can be used to guarantee consistency of joint actions in an economy e .

We conclude this section by observing the following. Consistent joint actions in an economy e imply the formation of coalitions with respect to specific activities. It is not necessary that, because a particular coalition has been formed with respect to one activity, the members of such coalition will go on and carry every activity in the set J as members of the same coalition. Although this possibility

cannot be excluded, depending on the aggregate technology of the economy one could find, quite easily, examples where to require that the members of a given coalition carry on more than a single activity will be inefficient.

A case in point is the example presented in the introduction of this chapter. Although that example cannot be used as a general equilibrium model, nonetheless, it reveals what coalition structures the three individuals must form for attaining Pareto optimal allocations, i.e., with respect to the activity of "producing potatoes" the relevant coalition structure is $\{\{A, B\}, \{C\}\}$, where Mr. C devotes no resources for this activity. On the other hand, with respect to the activity of "transportation and marketing of potatoes" the relevant coalition structure is $\{\{A, C\}, \{B\}\}$, where Mr. B devotes no resources for this activity. (Presumably, in this example, Mr. B, in his agreement with Mr. A, receives part of the money that Mr. A could get from the sale of his potatoes.)

In general then, we can justify our claim that to require members of a given coalition to carry on collectively more than a single activity may be inefficient, by saying that some people perform certain tasks quite well with a given group of people but they are not as good in performing other tasks with the same group of people. Hence, to allow coalitions to be formed with respect to specific activities, which is the case for consistent joint actions in an economy $\underline{e}$, takes care of such situations. In turn, this presents an advantage of our model of an economy over other models of economic environments that do not have this property.

Despite this advantage, an economic environment $\underline{e}$ has also its limitations in serving as a model that could explain certain phenomena. In particular, and as it is revealed from Lemma 2.1, above, the assumption that, in an economy $\underline{e}$, an individual cannot carry on, simultaneously, an activity as a member of more than a single coalition, does impose certain restrictions on the choices available to each individual $i \in N$. As an example, let us suppose that "production of commodity x " is one of the activities included in the set J . Then, in a (consistent) joint action an individual who decides to take part in the production of commodity x , contributing, say, his "labor", cannot do so, simultaneously, as a member of two different coalitions. In other words, and in such case, we could say that an economic environment $\underline{e}$ cannot serve as a model for situations where "full time and part time employment" in two different firms that produce the same commodity is possible. Nonetheless, we should point out that limitations of this kind exist, because, in this example, we have chosen the "production of commodity x " as a single activity without taking into account location and time factors. It may be possible then that most of these limitations could be removed by considering a sufficiently large set of activities J .

2.4 Coalition Actions

Given an economic environment $\underline{e}$, the set of joint actions A contains elements which, in the sense described in the preceding section, admit certain coalitions from the set C_N . In particular, we have defined such admissibility of coalitions by joint actions independently of the process that the set of individuals N , in an economy $\underline{e}$, follow, in selecting their actions with respect to the various activities in the set J . An implication of this will be the

following. Given a joint action $\underline{a}$, $\underline{a} \in A$, which admits a non-empty coalition c , $c \in C_N$, with respect to some activity j , $j \in J$, the corresponding, to such joint action, vector of actions $(\underline{a}_j^i)_{i \in c}$, where $\underline{a}_j^i = (k_j^i; c)$, $\forall i \in c$, does not, necessarily, represent an agreement among the members of coalition c . In other words, because the action $\underline{a}_j^i$, of each individual $i \in c$, has the common coalition component c , in the joint action $\underline{a}$, it is not necessary that the vector $(\underline{a}_j^i)_{i \in c}$ is the result of an agreement among the members of coalition c . Apparently, each element $\underline{a}_j^i$, of that vector, could have been selected by the respective individual $i \in c$ without any communication with the other members of coalition c . Hence, it could be just by coincidence that the coalition component of each action $\underline{a}_j^i$, as above, is equal to c and not as a result of some agreement among the members of that coalition. However, this does not preclude the possibility that the set of individuals N , of an economy $\underline{e}$, in selecting their actions with respect to the various activities in the set J , follow a process where vectors of the form $((k_j^i; c))_{i \in c}$, any $j \in J$, and any $c \in C_N$, could be interpreted as the result of an agreement among the members of the respective coalitions. The interpretation of such vectors of actions under such processes, and some of the implications that follow that interpretation, is the subject of this section.

Given an economic environment $\underline{e}$, we shall say that the set of individuals N follow a cooperative process in selecting their actions with respect to the various activities in the set J , if the following holds true. No individual $i \in N$ can decide on his own that he will carry on collectively with others an activity if there is not a prior agreement between him and those other individuals to that effect.

As an example, the bargaining process that takes place in a cooperative game constitutes a cooperative process.

In terms of actions we can interpret a cooperative process in an economy $\underline{e}$ as follows. Any vector of actions $((\ell_j^i; c_j^i))_{i \in c, j \in J}$, $c \in C_N$, $c \neq \emptyset$, that satisfies the condition $c_j^i = c$, $\forall i \in c$, can, always, be considered the result of an agreement among the members of the set c to the effect that those individuals have decided to carry on collectively activity j .

Using this interpretation, and without having to be specific about the particular cooperative process that the set of individuals N , in an economy $\underline{e}$, follow, in selecting their actions with respect to the various activities in the set J , we can say the following. Under a cooperative process, for any activity $j \in J$, and for any non-empty coalition $c \in C_N$, a vector of actions $(\underline{a}_j^i)_{i \in c}$, $\underline{a}_j^i = (\ell_j^i; c_j^i)$, $i \in c$, that satisfies the condition:

$$\underline{a}_j^i = (\ell_j^i; c) \quad , \quad \forall i \in c,$$

constitutes a "coalition action." In other words, in the sense that the set of individuals N follows a process where, any decision to carry on collectively some activity $j \in J$, by any set of individuals $c \in C_N$, requires an agreement to that effect; vectors $(\underline{a}_j^i)_{i \in c}$, that satisfy the above condition, must reflect that agreement. Hence, they constitute "coalition actions."

Given an economic environment $\underline{e}$, let us denote by $\underline{a}_j^c$, any $c \in C_N$, $c \neq \emptyset$, and any $j \in J$, the vector of actions with respect to activity j of the members of coalition c which have as a common coalition component that coalition. That is; let

$$(2.1) \quad \underline{a}_j^c \equiv ((x_j^i; c))_{i \in c}, \quad \forall c \in C_N, c \neq \emptyset, \text{ and } \forall j \in J.$$

From now on, we shall refer to a vector of actions $\underline{a}_j^c$, as above, as a coalition action, in particular, a coalition action, with respect to activity j, of coalition c, if and only if the set of individuals N, in the economy e , follow a cooperative process in selecting their actions with respect to the various activities in the set J.

Apparently, in terms of vectors of actions in the form given in (2.1), to say that a joint action $\underline{a} \in A$ admits a non-empty coalition $c \in C_N$ with respect to activity j, $j \in J$, means the following. The corresponding to $\underline{a}$ actions $\underline{a}_j^i$ of each individual $i \in c$, can be written in the form of a vector $\underline{a}_j^c$ as in (2.1), above. Therefore, and in terms of a cooperative process, the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$, associated with any joint action $\underline{a} \in A$, will represent all those coalitions which have reached an agreement about coalition actions, given $\underline{a}$.

This last conclusion becomes very important if one is to consider the set of consistent joint actions A^* of an economy e .

In particular, let us observe that, by using (2.1), above, in conjunction with Lemma 2.1, of the preceding section, we can express each consistent joint action $\underline{a} \in A^*$, solely, in terms of the vectors $\underline{a}_j^c$ corresponding to the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$. This, will amount to a reordering of the elements of each $\underline{a} \in A^*$. Hence, and denoting by $\rho(\cdot)$ such reordering,

$$(2.2) \quad \rho(\underline{a}) \equiv ((\underline{a}_j^c)_{c \in \tau_j})_{j \in J}, \text{ for each } \underline{a} \in A^*,$$

where, given $\underline{a} \in A^*$, τ_j represents the coalition structure that, according to Lemma 2.1, must be formed with respect to the respective activity j , each $j \in J$, because $\underline{a}$ is a consistent joint action.

In terms of a cooperative process, we can interpret $\rho(\underline{a})$, any $\underline{a} \in A^*$, as follows. If the set of individuals N , in an economy $\underline{e}$, follow a cooperative process in selecting their actions with respect to the various activities in the set J , and if, by so doing, they decide to undertake a joint action $\underline{a} \in A^*$, then, $\rho(\underline{a})$ will represent all the coalition actions decided upon by all coalitions which will be formed through such process, that is, the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$. In other words, for each $\underline{a} \in A^*$, $\rho(\underline{a})$ represents all the collective agreements that could be reached among the set of individuals N , with respect to each and every activity $j \in J$, given some cooperative process.

Apparently, consistent joint actions in the set A^* are not necessarily feasible. Therefore, even if the set of individuals N , in an economy $\underline{e}$, were to reach the collective agreements represented by the reordering $\rho(\underline{a})$ of some consistent joint action $\underline{a} \in A^*$, it is not necessary that such agreements could be carried out. However, this will not be the case if the set of individuals N were to decide on a feasible consistent joint action $\underline{a} \in \hat{A}^*$. Hence, it is feasible consistent joint actions $\underline{a} \in \hat{A}^*$ which represent the greatest interest as far as cooperative processes are concerned. For example, if $\underline{a} \in \hat{A}^*$, then $\rho(\underline{a})$ represents, not only all the collective agreements that could be reached among the set of individuals N , with respect to each and every activity $j \in J$, but collective agreements that could be carried out as well, given some cooperative process.

We can summarize this section as follows:

1. Given an economic environment $\underline{e}$, we can consider vectors of actions of the form $\underline{a}_j^c$, defined in (2.1), $j \in J$, $c \in C_N$, as coalition actions (i.e., as the result of an agreement, reached among the members of each set $c \in C_N$, to the effect that they will carry on collectively activity j) provided that we introduce a cooperative process which the set of individuals N follow in selecting their actions with respect to the various activities in the set J .

2. A cooperative process for an economy $\underline{e}$ will be well defined if it excludes the possibility that the set of individuals N could use a joint action which is not consistent.

3. To ensure that all coalition actions in the reordering $\rho(\underline{a})$ of a consistent joint action $\underline{a} \in A^*$ can be carried out, a cooperative process must, also, exclude the possibility that the set of individuals N , in an economy $\underline{e}$, could use a joint action which is not feasible.

Apparently, and in view of these conclusions, if we were to assume that the set of individuals N , in an economy $\underline{e}$, play a cooperative game where they could use only feasible consistent joint actions as "joint strategies", then we could say that the set of individuals N follow a cooperative process that has all of the above characteristics. This we proceed to do in the next section.

2.5 The Cooperative Game $\Gamma(\underline{e})$

In this section we show that an economic environment $\underline{e}$ is conceptually equivalent to an n -person cooperative game where each individual $i \in N$, as a player, could use strategies which imply that he is a member of more than a single coalition, simultaneously.

Let us start out by establishing, first, the equivalence of an economy e to an n -person game in normal form and then proceed to see in what sense such game can be reduced to a cooperative game with the above characteristic.

To describe a game in normal form we need the following information.

1. A set of players.
2. For each player in that set
 - a. a set of strategies,
 - b. an outcome function, and
 - c. a preference ordering over the set of outcomes that could guide that player in the selection of an optimal strategy, given the strategies used by the other players.

We claim that with the appropriate interpretation of elements in the 8-tuple $(N, \omega; Q; J, A, \psi; \bar{Y}, R)$, that describes an economic environment e , we can obtain a game in normal form, as above. Therefore, and in the sense that a game in normal form can be described, solely, from the information contained in the 8-tuple $(N, \omega; Q; J, A, \psi; \bar{Y}, R)$ that describes an economy e , the two will be conceptually equivalent to each other.

Given an economic environment e , and in order to describe a particular game in normal form, let us postulate the following.

1. The set of individuals N , in the economy, will represent the set of players for that game.
2. For each individual (player) $i \in N$, the set of feasible individual actions $\hat{A}^i$ will represent that individual's set of strategies in that game.

3. The aggregate production correspondence ψ will represent the outcome function of that game, in a sense that we will describe below.

4. For each individual (player) $i \in N$, the preference ordering $\succeq_i$, defined over a consumption set that includes the set of feasible final allocations (outcomes that result from the outcome function of the game) will be used to guide that individual in the selection of an optimal individual action (strategy) given the individual actions (strategies) of the others.

On the basis of the above four postulates then, $G(\underline{e})$, where

$$(2.3) \quad G(\underline{e}) \equiv (N; \hat{A}; \psi; \bar{Y}, R),$$

is an n -person game in normal form, relative to the economic environment $\underline{e}$ on the basis of which (economic environment) such game is defined.

Let us clarify now in what sense the aggregate production correspondence of an economy $\underline{e}$ serves as an outcome function for the game $G(\underline{e})$. By definition, the aggregate production correspondence ψ maps joint actions in the set A into the n -fold Cartesian product of the commodity space, i.e., $\psi: A \rightarrow Q^{(n)}$. Furthermore, for any $\underline{a} \in A$, the elements in $\psi(\underline{a})$ are final outcomes (allocations). Therefore, the restriction of ψ over the set of feasible joint actions $\hat{A}$ could be interpreted as a map from the set of joint strategies into a set of final outcomes. It is this restriction of ψ that we use as the outcome function of the game $G(\underline{e})$.

As far as each player $i \in N$ is concerned, a different kind of restriction of ψ is used for that player's outcome function. In particular, for any $i \in N$, and for any feasible joint action $\underline{a} \in \hat{A}$, let us write $\underline{a}$ as $(\underline{a}^i; \underline{a}^{-i})$, where $\underline{a}^i$ denotes the vector of individual

actions, relative to $\underline{a}$, corresponding to the set of individuals $N - \{i\}$, i.e., if for any individual $i \in N$ we were to define

$$(2.4) \quad \hat{A}^i \equiv \prod_{j \in (N - \{i\})} \hat{A}_j,$$

then $\underline{a}^i \in \hat{A}^i$, and $\hat{A} = \hat{A}^i \times \hat{A}^i$. Using this notation we can write the outcome function of the game $G(\underline{e})$ as $\psi(\underline{a}^i; \underline{a}^i)$, for any $i \in N$ and for any $\underline{a} \in \hat{A}$. Suppose now that we were to treat the individual actions (strategies) $\underline{a}^i$ undertaken by all players in $N - \{i\}$, some $i \in N$, as a given parameter, and allow variations only in the individual action (strategy) $\underline{a}^i$ over the set $\hat{A}^i$. Then, and with such parametric treatment of the strategies of the other players, the outcome function of the game $G(\underline{e})$ becomes the outcome function of the player whose strategy is not treated parametrically. In other words, for any given $\underline{a}^i \in \hat{A}^i$, the restriction of ψ over the set of feasible joint actions in $\hat{A}$ where only their i -th element is allowed to vary, represents the i -th player's outcome function of the game $G(\underline{e})$.

Let us proceed now to see in what sense the n -person game in normal form $G(\underline{e})$, defined above, could be reduced to an n -person cooperative game where each individual (player) $i \in N$ could use strategies which imply that he is a member of different coalitions. Thus, let us proceed to see in what sense we can establish an equivalence between such game and an economic environment $\underline{e}$.

Like any game in normal form, the game $G(\underline{e})$, defined above, "becomes cooperative if we allow the players to communicate before each play and to make binding agreements about the strategies they will use" (Aumann [3, p. 3]). Therefore, an economic environment $\underline{e}$ will be conceptually equivalent to an n -person cooperative game if, in the game

in normal form $G(\underline{e})$, we allow the set of individuals (players) N to communicate and make binding agreements about the actions that they will use with respect to the various activities in the set J .

Given an economic environment $\underline{e}$, let

$$(2.5) \quad \Gamma(\underline{e}) \equiv (N; \hat{A}^*; \psi; \bar{Y}, R).$$

We claim that if the players in a game $G(\underline{e})$ are allowed to communicate and make binding agreements about the strategies they will use, these players cannot reach binding agreements that will result in a joint strategy outside the set of feasible consistent joint actions $\hat{A}^*$.

Hence, we claim that the following theorem holds true.

Theorem 2.1: The 5-tuple $\Gamma(\underline{e})$, defined in (2.5), is an n -person cooperative game relative to a given economic environment $\underline{e}$.

Proof: Because the 5-tuple $\Gamma(\underline{e})$ is derivable from the 5-tuple that describes the game in normal form $G(\underline{e})$ (in fact, in form, the only difference between the two 5-tuples $G(\underline{e})$ and $\Gamma(\underline{e})$ is on the set of joint actions allowed as "joint strategies" in each) we only need to show the following. If the set of players N in the game $G(\underline{e})$ are allowed to communicate and make binding agreements, then, the selection of a joint action $\underline{a}$, $\underline{a} \in A$, such that $\underline{a} \notin \hat{A}^*$, as a joint strategy by those players, will lead to a contradiction.

We prove this in two steps.

1. (Consistency): Suppose that, under the above assumption, the set of players N were to select, as a joint strategy, a joint action $\underline{a}$, $\underline{a} \in A$, such that $\underline{a} \notin \hat{A}^*$.

From Section 2.3, above, if $\underline{a}$ is not a consistent joint action, the following holds true. There exist two different individuals, say individual i , and individual s , $i, s \in N$, and there exists an activity j , $j \in J$, such that, relative to $\underline{a}$, $\underline{a}_j^i = (\ell_j^i; c_j^i)$, and $\underline{a}_j^s = (\ell_j^s; c_j^s)$, are the actions with respect to activity j of the individuals i and s , respectively, and $s \in c_j^i$ but $c_j^i \neq c_j^s$. In other words, if $\underline{a}$ is not a consistent joint action then, and relative to $\underline{a}$, some player i , $i \in N$, undertakes an action $\underline{a}_j^i$, $\underline{a}_j^i = (\ell_j^i; c_j^i)$, with respect to some activity j , $j \in J$, which contains a coalition component c_j^i that has not been agreed upon by at least one other member of c_j^i , say player s . But then, player i could not have undertaken such action as a result of communication and a binding agreement between him and the other players in c_j^i . This, establishes the contradiction.

2. (Feasibility): Suppose that under the above assumption, the set of players N were to select, as a joint strategy, a joint action $\underline{a}$, $\underline{a} \in A$, which is consistent ($\underline{a} \in A^*$) but not feasible ($\underline{a} \notin \hat{A}$).

Apparently, the selection of a joint action, as above, as a joint strategy by the set of players N , will violate the rules of the game $G(\underline{e})$. In other words, it is in the game in normal form $G(\underline{e})$ where the set of players N are allowed to communicate and make binding agreements about the strategies they use. This, could restrict the joint strategies they can select, but not expand them. Hence, the contradiction is established because, by definition, the set of joint strategies allowed to the set of players N , in a game $G(\underline{e})$, consists of the set of feasible joint actions $\hat{A}$ of the respective economy $\underline{e}$.

Note that, independently of the violation of the rules of the game $G(\underline{e})$, but still within the context of such game in normal form, we can show that $\underline{a} \notin \hat{A}$ is a contradiction as follows. Given that communication and binding agreements, among the set of players N , in a game $G(\underline{e})$, cannot lead to the use of a joint action $\underline{a} \in A$ such that $\underline{a} \notin \hat{A}^*$ (see the part of the proof about consistency, above) if $\underline{a} \in \hat{A}^*$ but $\underline{a} \notin \hat{A}$ the following will hold true. Some players will not be able to fulfill their assumed responsibilities in $\underline{a}$. But then, the relevant agreements in such joint actions cannot be binding.

q.e.d.

It is of interest to observe here that Theorem 2.1, above, does not depend for its validity on the particular game, in normal form, $G(\underline{e})$. In fact, the reason, that we have included an alternative proof in establishing a contradiction if the set of players N were to select a joint action $\underline{a}$, $\underline{a} \notin \hat{A}$, as a joint strategy, was to show that any game in normal form, that does not necessarily, guarantee feasibility, could also serve our purposes, that is, a game in normal form, say $G'(\underline{e})$, that could differ from the game $G(\underline{e})$ only in the set of joint actions allowed as joint strategies to the set of players N , as long as such set includes, as a subset, the set of feasible consistent joint actions $\hat{A}^*$ of the relevant economy $\underline{e}$. For example, a game in normal form, where the entire set of joint actions A , of an economy $\underline{e}$, is used as the set of joint strategies for the set of players N , could be used here instead of the game $G(\underline{e})$, without any effects on the validity of Theorem 2.1.

An important implication of Theorem 2.1 is the following. Consistency of joint actions in an economy $\underline{e}$ can be attained by allowing the respective set of individuals N to play a cooperative game like $\Gamma(\underline{e})$. In turn, this implies that, through such game, the set of individuals N , in an economy $\underline{e}$, can reach binding agreements (through the selection of a joint action $\underline{a} \in \hat{A}^*$) where a different coalition structure could be formed with respect to different activities in the set J (see Lemma 2.1). But in a game $\Gamma(\underline{e})$, this can happen only if the players use strategies which imply that they belong to different coalitions, simultaneously.

The fact that players in a game $\Gamma(\underline{e})$ could use such strategies raises the question of what should we consider as "strategies for coalitions" in such games. We undertake the task of answering this question in the next section.

2.6 Strategies for Coalitions in a Game $\Gamma(\underline{e})$

A great part of the analysis in Section 2.4, above, about coalition actions, was based on the assumption that the set of individuals N , in an economy $\underline{e}$, could follow some cooperative process in selecting their actions with respect to the various activities in the set J . Apparently, and as we have pointed out at the end of that section, a cooperative game could be used to define such process. In particular then, we can use the bargaining process that takes place in a cooperative game $\Gamma(\underline{e})$ for that purpose. That is, we can proceed in this section by assuming that the set of individuals N , in an economy $\underline{e}$, play the cooperative game $\Gamma(\underline{e})$.

The first implication that follows from this assumption is that vectors of actions $\underline{a}_j^c$, defined in (2.1), compatible with feasible consistent joint actions in the set $\hat{A}^*$ in the sense that

$$(2.6) \quad \exists \underline{a}, \underline{a} \in \hat{A}^*, \quad \exists: \underline{a}_j^c \in \rho(\underline{a}), \quad (*)$$

can, always, be considered as coalition actions in a game $\Gamma(\underline{e})$. Hence, the reordering $\rho(\underline{a})$ of any joint strategy $\underline{a} \in \hat{A}^*$, in a game $\Gamma(\underline{e})$, consists of all coalition actions corresponding to the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$. But then, we can say that, if the set of individuals N , in an economy $\underline{e}$, through the bargaining process of the game $\Gamma(\underline{e})$ (i.e., by playing that game) were to make binding agreements that result in the use of some joint action $\underline{a}$, $\underline{a} \in \hat{A}^*$, as a joint strategy, then, the coalition actions in the vector $\rho(\underline{a})$ must represent the "strategies" used by all coalitions that will be formed through such process, that is, the set of $\underline{a}$ - admissible coalitions $C(\underline{a})$.

An immediate consequence of the above conclusion is that, in a game $\Gamma(\underline{e})$, central to the strategy used by any non-empty coalition in the set C_N , must be some coalition action, in particular, a coalition action compatible with a feasible consistent joint action in the set $\hat{A}^*$. Furthermore, and in the sense that a strategy by any non-empty coalition in the set C_N should reflect a cooperative effort by the members of that coalition, strategies for non-empty coalitions in a game $\Gamma(\underline{e})$ must consist of one or more coalition actions with respect to the various

(*) Here, we use the notation: $\underline{a}_j^c \in \rho(\underline{a})$, for any $c \in C(\underline{a})$, for any $j \in J$, and for any $\underline{a} \in \hat{A}^*$, in order to indicate that $\underline{a}_j^c$ is an element of the vector $\rho(\underline{a})$.

activities in the set J . For example, and as is revealed from the reordering $\rho(\underline{a})$ of any joint action $\underline{a} \in \hat{A}^*$, the corresponding to some coalition $c \in C(\underline{a})$, coalition actions, could range, from a coalition action with respect to a single activity $j \in J$, up to a coalition action with respect to each and every activity in the set J . This implies that in a game $\Gamma(\underline{e})$ we can start out by designating single coalition actions as strategies for the respective coalitions, albeit as strategies which do not, necessarily, exhaust all the strategic possibilities of all players in the respective coalitions.

For each non-empty coalition $c \in C_N$, and for each activity $j \in J$, let $\hat{A}_j^{*c}$ denote the set of all coalition actions available to coalition c with respect to activity j in a game $\Gamma(\underline{e})$. That is, let

$$(2.7) \quad \hat{A}_j^{*c} \equiv \{\underline{a}_j^c = ((\ell_j^i; c))_{i \in c} : \underline{a}, \underline{a} \in \hat{A}^*, c \in C(\underline{a}),$$

$$\text{and } \underline{a}_j^c \in \rho(\underline{a})\}.$$

Apparently, elements in the set $\hat{A}_j^{*c}$ may differ from each other only on the amount of resources (input) devoted to activity j by individual members of coalition c .

For each non-empty coalition $c \in C_N$, let $\hat{A}^{*c}$ denote the set of all coalition actions available to coalition c with respect to each and every activity $j \in J$, in a game $\Gamma(\underline{e})$, i.e., let,

$$(2.8) \quad \hat{A}^{*c} \equiv \bigcup_{j \in J} \hat{A}_j^{*c}.$$

Finally, let us define the correspondence ξ , where

$$(2.9) \quad \xi: \begin{array}{l} U \quad \hat{A}^{*c} \rightarrow \hat{A}^*, \\ c \in C_N \\ c \neq \emptyset \end{array}$$

as follows

$$(2.10) \quad \xi(\underline{a}_j^c) \equiv \{\underline{a} \in \hat{A}^* : \underline{a}_j^c \in \rho(\underline{a})\}.$$

In words, $\xi(\underline{a}_j^c)$ is the set of feasible consistent joint actions in $\hat{A}^*$ such that the coalition action $\underline{a}_j^c$, $c \in C_N$, $c \neq \emptyset$, $j \in J$, is obtainable through a reordering of their elements.

Using the above notation and definition, we can say that, in a game $\Gamma(\underline{e})$, a coalition action $\underline{a}_j^c$, $\underline{a}_j^c \in \hat{A}^{*c}$, some $j \in J$, constitutes a strategy for coalition c , $c \in C_N$, in the following sense. Whenever the members of coalition c reach an agreement about carrying on collectively activity j by undertaking the coalition action $\underline{a}_j^c$, then, these players will not pursue to make agreements regarding coalition actions with respect to activities in the set J that cannot lead to some feasible consistent joint action $\underline{a} \in \xi(\underline{a}_j^c)$. This, could also be stated as saying that once the members of coalition c have decided on the coalition action $\underline{a}_j^c$ then they are free to pursue and make agreements about coalition actions with respect to each and every activity in the set J , different than activity j , as long as such coalition actions are compatible with some feasible consistent joint action $\underline{a} \in \xi(\underline{a}_j^c)$. (Note that this does not preclude the possibility that those additional agreements could be made among the members of coalition c . However, it does not require that these agreements will be confined among the members of coalition c , either.)

The use of single coalition actions as strategies for non-empty coalitions in a game $\Gamma(\underline{e})$ suggests that we can interpret the bargaining process of a game $\Gamma(\underline{e})$ as a two-step process in the following way.

Step 1: The members of any non-empty coalition c , $c \in C_N$, get together and, at least initially, they agree that they will carry on, collectively, activity j , some $j \in J$, and they will do so by having each member $i \in c$ devote the amount of resources ℓ_j^i to that activity. This amounts to coalition c deciding, initially, to undertake a coalition action $\underline{a}_j^c$.

Step 2: Given that the agreement represented by the coalition action $\underline{a}_j^c$, as above, does not exhaust every member's resources, each player $i \in c$, with resources left over, is free to pursue and make agreements with others (i.e., with other members of coalition c , or with players from $(N-c)$) as far as his resources permit and as far as activities different than activity j are concerned, that is, as long as such agreements are compatible with a joint action $\underline{a} \in \xi(\underline{a}_j^c)$.

Let us observe now that if this two-step bargaining process is to reach some conclusion where a feasible consistent joint action $\underline{a} \in \hat{A}^*$ will be used as a joint strategy by the set of players N , then the members of any coalition $\tilde{c} \in C(\underline{a})$ must be involved in agreements, represented by coalition actions in $\rho(\underline{a})$, with respect to each and every activity $j \in J$. That is, given a joint action $\underline{a}$, as above, in addition to the coalition action that an $\underline{a}$ - admissible coalition uses as a strategy, the members of such coalition must be involved in other coalition actions as well. (This follows from Lemma 2.1, in conjunction with the fact that elements of the set $\hat{A}^*$ are consistent joint actions.)

But then, simple coalition actions used as strategies for admissible coalitions in the set $C(\underline{a})$ may not provide sufficient information as to the effectiveness of such coalitions. For example, all that the set $\xi(\underline{a}_j^{\tilde{c}})$ reveals, is that if coalition $\tilde{c}$ uses as a strategy the coalition action $\underline{a}_j^{\tilde{c}}$ then it can expect that some joint action $\underline{a} \in \xi(\underline{a}_j^{\tilde{c}})$ may result as a joint strategy by the set of players N . Nonetheless, the members of such coalition participate in coalition actions with respect to each and every activity in the set J . Therefore, at times, it may be more useful to include all the actions that the members of coalition $\tilde{c}$, as above, undertake in addition to the coalition action $\underline{a}_j^{\tilde{c}}$, in the strategy of that coalition.

To see how this can be done and in what sense a non-empty coalition $\tilde{c} \in C_N$ could use such strategies, let us observe that the reordering $\rho(\underline{a})$ of any feasible consistent joint action $\underline{a} \in \hat{A}^*$, which by (2.2) we can write, in terms of coalition actions, as

$$\rho(\underline{a}) = ((\underline{a}_j^c)_{c \in \tau_j})_{j \in J},$$

contains coalition actions $\underline{a}_j^c$ that either involve members of coalition $\tilde{c}$, some $\tilde{c} \in C(\underline{a})$, or do not involve such members. Hence, for each $\underline{a} \in \hat{A}^*$, for each $\tilde{c} \in C(\underline{a})$, and for each $j \in J$, we can define the set of coalitions $\tau_j(\tilde{c})$ by

$$(2.11) \quad \tau_j(\tilde{c}) \equiv \{c \in \tau_j : c \cap \tilde{c} \neq \emptyset\}.$$

In words, relative to a joint action $\underline{a} \in \hat{A}^*$, the set $\tau_j(\tilde{c})$ consists of all coalitions in the coalition structure τ_j that contain at least one player from coalition $\tilde{c}$. Furthermore, what holds true about the sets

$\tau_j(\tilde{c})$ is that if coalition $\tilde{c}$ is an $\underline{a}$ - admissible coalition with respect to activity s , then for each such activity, $\tau_s(\tilde{c}) = \{\tilde{c}\}$.

The definition of the sets $\tau_j(\tilde{c})$, above, permits us to reorder the elements of $\rho(\underline{a})$ so that, all coalition actions in $\rho(\underline{a})$ undertaken by coalitions which contain at least one member of coalition $\tilde{c}$ will appear together. That is, for each $\underline{a} \in \hat{A}^*$, and for each $\tilde{c} \in C(\underline{a})$, let

$$(2.12) \quad \underline{a}(\tilde{c}) \equiv ((\underline{a}_j^C)_C \in \tau_j(\tilde{c}))_{j \in J},$$

and let

$$(2.13) \quad \underline{a}(N-\tilde{c}) \equiv ((\underline{a}_j^C)_C \in (\tau_j - \tau_j(\tilde{c}))_{j \in J}^*.$$

Then, this reordering of the elements of $\rho(\underline{a})$ which we will denote by $\rho^{\tilde{c}}(\underline{a})$, for each $\tilde{c} \in C(\underline{a})$, takes the form

$$(2.14) \quad \rho^{\tilde{c}}(\underline{a}) \equiv (\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c})).$$

Let us consider now any vector of coalition actions $\underline{a}(\tilde{c})$, some $\tilde{c}, \tilde{c} \in C_N, \tilde{c} \neq \emptyset$, which is compatible (in the sense of (2.14), above) with some feasible consistent joint action $\underline{a} \in \hat{A}^*$. In addition to the fact that a vector of coalition actions $\underline{a}(\tilde{c})$, as above, contains all coalition actions that involve members of coalition $\tilde{c}$, given a joint action $\underline{a}, \underline{a} \in \hat{A}^*$, such that (2.14) is satisfied, the following hold true about the vector $\underline{a}(\tilde{c})$.

1. Central to such vector of coalition actions, there is at least one coalition action by coalition $\tilde{c}$, say the coalition action $\underline{a}_j^{\tilde{c}}$, $\underline{a}_j^{\tilde{c}} \in \hat{A}^*$.

2. The actions of individual players with respect to different activities in the set J that comprise the coalition actions in $\underline{a}(\tilde{c})$ can

be separated into two different groups. The first group will contain the actions with respect to each and every activity in the set J of each player $i \in \tilde{c}$, i.e., it consists of the elements in the corresponding vector of individual actions $(\underline{a}^i)_i \in \tilde{c}$. The second group will contain all those actions of players in $(N-\tilde{c})$ which, combined with actions in $(\underline{a}^i)_i \in \tilde{c}$, yield the respective coalition actions, in the vector $\underline{a}(\tilde{c})$, of coalitions that contain players from both sets $\tilde{c}$ and $(N-\tilde{c})$. (Apparently, for a given $\underline{a}(\tilde{c})$ this latter group of actions could be empty.)

With the above notation and observations in mind, let us suppose that coalition $\tilde{c}$ starts out by using as a strategy the coalition action $\underline{a}_{\tilde{c}}, \underline{a}_{\tilde{c}} \in \hat{A}^{\tilde{c}}$. This, will not, necessarily, put an end to the strategic behavior of the members of coalition $\tilde{c}$, because each player $i \in \tilde{c}$ with resources left over is free to pursue and make agreements about coalition actions with respect to activities different than activity j . Hence, and if the bargaining process of the game $\Gamma(e)$ is to lead to some joint strategy $\underline{a}, \underline{a} \in \hat{A}^*$, such that $\underline{a}_{\tilde{c}} \in \rho(\underline{a})$, then the players of coalition $\tilde{c}$ must undertake the corresponding to such joint strategy, vector of individual actions $(\underline{a}^i)_i \in \tilde{c}$. But then we can go ahead and call such vectors of individual actions "strategies" for the respective coalitions.

The problem, with designating vectors of individual actions, as above, as "strategies" for the respective coalitions, is that it is not always true that the members of a given coalition $\tilde{c} \in C_N$ could go ahead and, independently of the actions of players in $(N-\tilde{c})$, use such vectors of individual actions as strategies, at will. For example, and independently of the actions of its complement, a coalition $\tilde{c} \in C_N$ could always go ahead and use some coalition action $\underline{a}_{\tilde{c}} \in \hat{A}^{\tilde{c}}$ as a strategy. On the other hand, and due to the rules of the game $\Gamma(e)$, the same

coalition $\tilde{c}$, in order to use a vector $(\underline{a}^i)_{i \in \tilde{c}}$ as a "strategy" may have to ensure that players in the set $(N - \tilde{c})$ are willing to make the agreements necessary for coalition actions, in the respective vector $\underline{a}(\tilde{c})$, that require their consent. Therefore, to say that a vector of individual actions $(\underline{a}^i)_{i \in \tilde{c}}$, as above, constitutes a "strategy" for coalition $\tilde{c}$ must be understood in a conditional sense. Thus, we can call such "strategies," conditional strategies, that is, strategies whose use by the respective coalitions may be conditional on the consent of players in the complement of such coalitions.

To be more precise about the conditionality of vectors of individual actions as strategies, let us use the notation introduced above. Given a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, given a coalition action $\underline{a}_{\tilde{c}} \in \hat{A}^{\tilde{c}}$, and given a joint action $\underline{a} \in \hat{A}^*$, such that $\underline{a}_{\tilde{c}} \in \rho(\underline{a})$, let us consider the reordering $\rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}); \underline{a}(N - \tilde{c}))$ of such joint action. Then, to say that the corresponding to such joint action vector of individual actions $(\underline{a}^i)_{i \in \tilde{c}}$ is a conditional strategy for coalition $\tilde{c}$ means the following. In order for coalition $\tilde{c}$ to use the vector $(\underline{a}^i)_{i \in \tilde{c}}$ as a strategy in the game $\Gamma(\underline{e})$, players in the set $(N - \tilde{c})$ must be willing to go along and undertake the corresponding actions with respect to coalition actions in the vector $\underline{a}(\tilde{c})$, as may be required.

Because an arbitrary vector of individual actions $(\underline{a}^i)_{i \in \tilde{c}}$ does not guarantee that the respective coalition $\tilde{c}$ does undertake some coalition action, even if such vector is compatible with a joint action $\underline{a} \in \hat{A}^*$ in the sense that $(\underline{a}^i)_{i \in \tilde{c}}$ represents the corresponding to $\underline{a}$ individual action of each player $i \in \tilde{c}$, but a vector of coalition actions $\underline{a}(\tilde{c})$, in the reordering $\rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}), \underline{a}(N - \tilde{c}))$, of any $\underline{a} \in \hat{A}^*$ such that $\tilde{c} \in C(\underline{a})$, does. Furthermore, and because a vector of coalition actions

$\underline{a}(\tilde{c})$, as above, reveals precisely what actions each player involved in coalition actions in that vector must undertake, but as far as players in $(N-\tilde{c})$ are concerned, the corresponding vector $(\underline{a}^i)_i \in \tilde{c}$ does not, from now on, and in order to avoid any confusion or misunderstandings we will proceed as follows. We will refer to a vector of coalition actions $\underline{a}(\tilde{c})$, as above, as a conditional strategy for coalition $\tilde{c}$, $\tilde{c} \in C_N$, in the game $\Gamma(\underline{e})$. Thus, to say that a vector of coalition actions $\underline{a}(\tilde{c})$, in the reordering $(\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c}))$ of some feasible consistent joint action $\underline{a} \in \hat{A}^*$, such that $\tilde{c} \in C(\underline{a})$, is a conditional strategy for coalition $\tilde{c}$, should be interpreted as meaning the following. First, coalition $\tilde{c}$ undertakes a coalition action, say the coalition action $\underline{a}_j^{\tilde{c}}, \underline{a}_j^{\tilde{c}} \in \hat{A}^{*\tilde{c}}$, and $\underline{a}_j^{\tilde{c}} \in \underline{a}(\tilde{c})$. Second, and provided that players in $(N-\tilde{c})$ are willing to go along and make the agreements that may be required by the coalition actions in $\underline{a}(\tilde{c})$, with players in $\tilde{c}$, coalition $\tilde{c}$ could use as a strategy the corresponding vector of individual actions $(\underline{a}^i)_i \in \tilde{c}$.

Apparently, conditional strategies, for a non-empty coalition $\tilde{c} \in C_N$, are not, necessarily, strategies that can be used at will by that coalition. For example, a coalition $\tilde{c}$ may never be able to use certain conditional strategies $\underline{a}(\tilde{c})$ because players in the set $(N-\tilde{c})$ will never agree to go along with the relevant coalition actions that require their consent. This is the reason that strategies $\underline{a}(\tilde{c})$ have been called conditional strategies, in the first place. The natural question to ask here, then, is whether and under what conditions we can say that a given coalition $\tilde{c} \in C_N$ could go ahead and use some of its conditional strategies.

Obviously, if the members of a given coalition limit their agreements about all coalition actions in the set J among themselves then, the

consent of players in the set $(N - \tilde{c})$ is completely unnecessary. Hence, any conditional strategy $\underline{a}(\tilde{c})$ that satisfies this condition, for some $\tilde{c} \in C_N$, can be used at will by that coalition. From now on we will refer to strategies $\underline{a}(\tilde{c})$ that satisfy this condition as "independent strategies" for the respective coalitions $\tilde{c}$. In other words, for each non-empty coalition $\tilde{c} \in C_N$, a vector of coalition actions $\underline{a}(\tilde{c})$ is an independent strategy for coalition $\tilde{c}$ if and only if the following holds true. There exists a joint action $\underline{a} \in \hat{A}^*$, such that (i) $\tilde{c} \in C(\underline{a})$, (ii) $\rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}); \underline{a}(N - \tilde{c}))$, (iii) $\underline{a}(\tilde{c}) = ((\underline{a}_j^c)_{c \in \tau_j(\tilde{c})})_{j \in J}$, and (iv) for each $j \in J$, and for each $c \in \tau_j(\tilde{c})$, what holds true is that $c \cap (N - \tilde{c}) = \emptyset$. In simpler terms, this is equivalent to saying that by reordering the elements of a vector of individual actions $(\underline{a}^i)_{i \in \tilde{c}}$ that correspond to a vector of coalition actions $\underline{a}(\tilde{c})$ we can obtain, exactly, the coalition actions in that vector.

Our approach of treating independent strategies as an extreme case of conditional strategies has some advantages, as we shall see in the next chapter. However, the question remains whether and under what conditions we can say that a given coalition $\tilde{c} \in C_N$ could go ahead and use some of its conditional strategies if such strategies do not constitute independent strategies in the sense described above.

To answer this question, for a given n-person cooperative game $\Gamma(e)$, let $Z(\Gamma(e))$ denote the set of all pairs $(\underline{a}; \underline{y})$, where $\underline{a}$ is a feasible consistent joint action, i.e., $\underline{a} \in \hat{A}^*$, and $\underline{y}$ is a final outcome that could result from such joint action, i.e., $\underline{y} \in \psi(\underline{a})$. Hence,

$$(2.15) \quad Z(\Gamma(e)) \equiv \{(\underline{a}; \underline{y}) : \underline{a} \in \hat{A}^*, \underline{y} \in \psi(\underline{a})\}.$$

We shall refer to any element $(\underline{a}; \underline{y})$ of the set $Z(\Gamma(\underline{e}))$ as an alternative in the game $\Gamma(\underline{e})$. Thus, $Z(\Gamma(\underline{e}))$ represents the set of alternatives for the game $\Gamma(\underline{e})$.

One possible interpretation of elements in the set $Z(\Gamma(\underline{e}))$ is the following. If the set of individuals N , in an economy $\underline{e}$, through the bargaining process of the game $\Gamma(\underline{e})$, i.e., by playing that game, are to reach binding agreements with respect to each and every activity $j \in J$, then they must face the prospect that (i) they will be using a joint action $\underline{a} \in \hat{A}^*$ as a joint strategy, and (ii) they will settle at a final outcome (allocation) $\underline{y}$ that results from such joint strategy. Thus, the set $Z(\Gamma(\underline{e}))$ represents the set of all alternatives that the set of individuals N , in an economy $\underline{e}$, must consider for making their binding agreements, if they play the cooperative game $\Gamma(\underline{e})$.

With this interpretation of elements in the set $Z(\Gamma(\underline{e}))$ in mind, we could say that whether or not, players in the set $(N-\tilde{c})$ will be willing to go along with the coalition actions in a vector $\tilde{a}(\tilde{c})$ that require their consent, may depend on the particular stage of the bargaining process of the game $\Gamma(\underline{e})$. That is, if the set of players N are considering an alternative $(\underline{a}; \underline{y})$, $(\underline{a}; \underline{y}) \in Z(\Gamma(\underline{e}))$, and if the question is whether a coalition $\tilde{c}$, $\tilde{c} \in C_N$, could use a conditional strategy $\tilde{a}(\tilde{c})$ that would change the alternative under consideration $(\underline{a}; \underline{y})$ to a new one, say to the alternative $(\tilde{a}; \tilde{y})$, then, and whether or not the relevant players in $(N-\tilde{c})$ will be willing to go along with the relevant coalition actions in $\tilde{a}(\tilde{c})$ that require their consent, could be determined by whether these players in $(N-\tilde{c})$ cannot expect to be worse off than what they could be by not doing so. Thus, to say that whether or not, players in the set $(N-\tilde{c})$ will be willing to go along with the coalition actions in a vector

$\tilde{a}(\tilde{c})$ that require their consent may depend on the particular stage of the bargaining process of the game $\Gamma(\underline{e})$, is equivalent to saying the following. Such consent may depend on a given alternative $(\underline{a}; \underline{y})$ which can be used as a starting point in our analysis.

This conclusion implies that a non-empty coalition $\tilde{c} \in C_N$, may be able to use some of its conditional strategies at some times but not at others.

Because in our analysis, in the next chapter, it will be important to know when a coalition $\tilde{c}$, $\tilde{c} \in C_N$, can use some of its conditional strategies, starting from a given alternative $(\underline{a}; \underline{y}) \in Z(\Gamma(\underline{e}))$, we introduce below the concept of the "consent correspondence of a game $\Gamma(\underline{e})$." For any non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, and for any alternative $(\underline{a}; \underline{y})$, $(\underline{a}; \underline{y}) \in Z(\Gamma(\underline{e}))$, we will denote by

$$\zeta(\tilde{c}; (\underline{a}; \underline{y}))$$

the set of conditional strategies $\tilde{a}(\tilde{c})$ for coalition $\tilde{c}$ such that either one of the following two conditions hold true.

1. The coalition actions in $\tilde{a}(\tilde{c})$ do not require the participation of any player in $(N - \tilde{c})$, i.e., $\tilde{a}(\tilde{c})$ is an independent strategy for coalition $\tilde{c}$.

2. Whenever, the participation of players in $(N - \tilde{c})$ is required for any coalition action in $\tilde{a}(\tilde{c})$, then those players in $(N - \tilde{c})$ do indeed agree to undertake the relevant actions necessary for the respective coalition actions in $\tilde{a}(\tilde{c})$.

Let us observe that, although we have not specified precisely what elements will be included in the set $\zeta(\tilde{c}; (\underline{a}; \underline{y}))$, still, we can say the following. A conditional strategy $\tilde{a}(\tilde{c})$ that belongs to the set

$\zeta(\tilde{c}; (\underline{a}; \underline{y}))$, is a strategy that can be used by coalition $\tilde{c}$ relative to the alternative $(\underline{a}; \underline{y})$. Because for a conditional strategy $\tilde{a}(\tilde{c})$ to belong to the correspondence $\zeta(\tilde{c}; (\underline{a}; \underline{y}))$, for a given $\tilde{c} \in C_N$, and a given $(\underline{a}; \underline{y})$, the consent of players in $(N-\tilde{c})$ is guaranteed, even if that consent is null (unnecessary), we refer to the correspondence $\zeta(\cdot)$ as the consent correspondence of a game $\Gamma(\underline{e})$.

2.7 The Cooperative Game $\tilde{\Gamma}(\underline{e})$

Given an economic environment $\underline{e}$, the cooperative game $\Gamma(\underline{e})$ is not the only cooperative game that defines a cooperative process which the set of individuals N could follow in selecting their actions with respect to the various activities in the set J . Our purpose in this section is to define an alternative cooperative game that the set of individuals N , in an economy $\underline{e}$, could be playing. In particular, a game where individual players cannot use strategies which imply that, simultaneously, they are members of more than a single coalition.

As we have pointed out in the preceding section, a set of individuals c , $c \in C_N$, who get together and decide, at least initially, to carry on collectively some activity $j \in J$, may also decide to make collective decisions about each and every activity in the set J without involving any individual from the set $(N-c)$. Although, this kind of collective decisions are voluntary in a game $\Gamma(\underline{e})$, that, may not always be the case. It is of interest then to consider a cooperative game where, in an economic environment $\underline{e}$, either by choice or due to some outside restrictions, the members of any non-empty coalition $c \in C_N$ must make collective decisions with respect to each and every activity in the set J without involving any individuals from the set $(N-c)$.

We start out denoting by $\tilde{A}$, $\tilde{A} \subseteq \hat{A}^*$, the set of feasible consistent joint actions in $\hat{A}^*$ which imply the formation of a common coalition structure, with respect to each and every activity in the set J , among the set of individuals N , in an economy $\underline{e}$. That is,

$$(2.16) \quad \tilde{A} \equiv \{ \underline{a} \in \hat{A}^* : \rho(\underline{a}) = ((\underline{a}_j^c)_{c \in \tau_j})_{j \in J}, \text{ and}$$

$$\exists \tau, \tau \in T_N, : \exists \tau_j = \tau, \forall j \in J \}.$$

Let us suppose now that the set of individuals N , in an economy $\underline{e}$, in selecting their actions with respect to the various activities in the set J , follow a cooperative process that could lead only to joint actions in the set $\tilde{A}$. This will be equivalent to imposing the following restriction on the cooperative process that the set of individuals N , in an economy $\underline{e}$, follow by playing the cooperative game $\Gamma(\underline{e})$. The set of individuals N , in an economy $\underline{e}$, in playing the cooperative game $\Gamma(\underline{e})$, are not permitted to use joint actions in the set $(\hat{A}^* - \tilde{A})$ as joint strategies.

Let,

$$(2.17) \quad \tilde{\Gamma}(\underline{e}) \equiv (N, \tilde{A}; \psi; \bar{Y}, R).$$

Then, the 5-tuple $\tilde{\Gamma}(\underline{e})$ represents the above restriction on the cooperative game $\Gamma(\underline{e})$. Therefore, and because the only difference between the 5-tuples $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$ is a restriction in joint strategies, of the former, the 5-tuple $\tilde{\Gamma}(\underline{e})$ constitutes a cooperative game relative to the respective economic environment $\underline{e}$.

The interesting thing about the cooperative games $\tilde{\Gamma}(\underline{e})$ is on the strategies that non-empty coalitions in the set C_N can use. In fact, and using the notation introduced in the preceding section, given any non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, any vector of coalition actions $\underline{a}(\tilde{c})$, such that $\rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c}))$, for any $\underline{a} \in \tilde{A}$, constitutes a strategy for coalition $\tilde{c}$. For that matter, such vectors of coalition actions constitute (in the terminology of the preceding section) independent strategies for the respective coalitions. In other words, and because, for any joint action $\underline{a} \in \tilde{A}$, and for any coalition $\tilde{c} \in C(\underline{a})$, the sets of coalitions $\tau_j(\tilde{c})$, relative to such joint actions, defined in (2.11), are such that $\tau_j(\tilde{c}) = \{\tilde{c}\}$, $\forall j \in J$, then the vector $\underline{a}(\tilde{c})$ in the reordering $\rho^{\tilde{c}}(\underline{a})$ of $\underline{a}$, takes the form: $\underline{a}(\tilde{c}) = (\underline{a}_j^{\tilde{c}})_{j \in J}$. Therefore, the consent of players in $(N-\tilde{c})$ is completely unnecessary for a coalition $\tilde{c}$ to use as a strategy a vector of coalition actions $\underline{a}(\tilde{c})$, as above.

The conclusion that follows from the above discussion is that, in a cooperative game $\tilde{\Gamma}(\underline{e})$, non-empty coalitions $\tilde{c} \in C_N$ could use only one kind of strategies and these strategies consist of vectors of coalition actions $\underline{a}(\tilde{c})$ in the reordering $\rho^{\tilde{c}}(\underline{a})$, of any joint action $\underline{a} \in \tilde{A}$. Furthermore, these are independent strategies for the respective coalitions, in the sense of the preceding section.

For a given economic environment $\underline{e}$, the n-person cooperative game $\tilde{\Gamma}(\underline{e})$ is more familiar in the economic literature than the game $\Gamma(\underline{e})$. More specifically, by assuming that the set of individuals in an economy $\underline{e}$ play the cooperative game $\tilde{\Gamma}(\underline{e})$ we can give the following interpretation of what non-empty coalitions can do in that economy. The members of any non-empty coalition $c \in C_N$, can pool together their (productive) resources and undertake a coalition action with respect to

each and every activity $j \in J$, but they cannot, simultaneously, undertake coalition actions with individuals outside their own coalition. Apparently, this interpretation, of what coalitions can do in an economy $\underline{e}$, is not valid if we were to assume that the set of individuals N play the cooperative game $\Gamma(\underline{e})$ and $(\hat{A}^* - \tilde{A}) \neq \emptyset$. Therefore, given an economy $\underline{e}$, the cooperative game $\tilde{\Gamma}(\underline{e})$ can be viewed as the cooperative game with the "built-in" restriction, mentioned in the introduction of this chapter.

Although what non-empty coalitions can do in an economy $\underline{e}$ is not necessarily invariant under the rules of each game $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, still, it is possible that for certain economic environments $\underline{e}$ there would not be any loss of generality if we were to assume that the respective set of individuals play the cooperative game $\tilde{\Gamma}(\underline{e})$ instead of the game $\Gamma(\underline{e})$. For example, if our concern is with specific solution concepts and these concepts are invariant under the rules of each game $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, for a given economic environment $\underline{e}$, then the use of the game $\tilde{\Gamma}(\underline{e})$ instead of the game $\Gamma(\underline{e})$ will not affect our analysis. Of course, for this to hold true, it will be required that the specific economic environment $\underline{e}$, or the particular solution concept that we use, be such that some sort of "strategic equivalence" can be established between the two games. Now, and as the example presented in the introduction of this chapter will indicate, such strategic equivalence will be impossible to establish for every economic environment $\underline{e}$. Therefore, to specify that in an economic environment $\underline{e}$ the set of individuals N play the cooperative game $\tilde{\Gamma}(\underline{e})$ regardless of whether a strategic equivalence between $\tilde{\Gamma}(\underline{e})$ and $\Gamma(\underline{e})$ can be established, is equivalent to ignoring a whole set of joint strategies that may be available to the set of individuals N and may be essential to our analysis.

2.8 Coalition Production Economies

In [14], and subsequently in [15], and [16] (see also D. Sonderman [41]) W. Hildenbrand introduced a class of economic environments, which he called coalition production economies, with the following characteristic. Technologies are assigned to groups of economic agents (coalitions) prior to the introduction of "firms" or "producers," as independent economic agents into the economy. Apparently, and in the sense that in an economic environment e (see relevant discussion in Section 2.2, above) we introduce a technology prior to the introduction of "firms" or "producers" into the economy, the class of coalition production economies and the class of economic environments e introduced in Section 2.2, above, share a common characteristic. Our purpose in this section is to compare those two classes of economic environments.

Let us recall from Section 2.2, above, that the reason that we decided to specify only an aggregate technology for an economy e is that we considered it to be the best way for dealing with externality phenomena of any kind. Furthermore, in an economic environment e , individual economic agents can carry on, collectively, different activities as members of different coalitions. But then, and in comparing the class of economic environments e with the class of coalition production economies we can say the following.

1. Coalition production economies with a finite set of economic agents are special cases of economic environments e .
2. An economic environment e is not always equivalent to a coalition production economy.

To elaborate on these two points let us observe the following. If we know the technology of each coalition then we can always obtain the

aggregate technology of an economy. Hence, given a coalition production economy where a technology has been assigned to each coalition, first we can obtain the set of outcomes that will result if the set of individuals form a particular coalition structure. This of course will not, necessarily, determine a set of allocations. But then we can consider all feasible allocations for each coalition, and hence obtain a set of final allocations for the economy, given a particular coalition structure. Apparently, this approach will yield an economic environment $\underline{e}$ where there will not be any difference between the sets $\hat{A}^*$ and $\tilde{A}$ because the assignment of technologies to coalitions, in a coalition production economy, excludes the possibility that individuals could, simultaneously, undertake activities as members of more than a single coalition. This implies that an economic environment $\underline{e}$ that has been derived from a coalition production economy, will have the following characteristic. The two games $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, introduced in the preceding two sections will be such that $\Gamma(\underline{e}) = \tilde{\Gamma}(\underline{e})$.

The above conclusion implies that a necessary condition, for obtaining a coalition production economy out of an economic environment $\underline{e}$, is that the set of individuals N should be restricted in their selection of joint actions to the set $\tilde{A}$. From this point on, i.e., after we impose the above restriction, the technology of each coalition $c \in C_N$ could be obtained from the aggregate production correspondence $\psi(\cdot)$. Of course some problems will be encountered for such derivation because in the presence of externalities in production it is not so clear what productions can be carried out by the members of a coalition $c \in C_N$ "independently of the actions of the other economic agents." But, in the presence of externalities, the same problem, i.e., of specifying the technology of

each coalition, is encountered in the definition of a coalition production economy (see, e.g., Sondermann [41, p. 264], especially footnote 6).

As a conclusion to the preceding discussion we could say the following. A coalition production economy could represent an economic environment $\underline{e}$ where the set of individuals N play the cooperative game $\tilde{r}(\underline{e})$.

CHAPTER THREE
CORE-CONCEPTS FOR AN ECONOMIC ENVIRONMENT $\underline{e}$

3.1 Introduction

By establishing a conceptual equivalence between models of economies and cooperative games, one can apply the tools of cooperative game theory to economic analysis. Solution concepts to cooperative games can be used to study stability properties of various allocations in an economy, and thus, predict the likelihood that economic agents would settle at one allocation or another. The "theory of the core" has been built on such premise.

In this chapter we study the problem of defining core-concepts for the two cooperative games introduced in the preceding chapter relative to an economic environment $\underline{e}$. Furthermore, we study the influence that the presence of externalities and public goods may have on those core-concepts.

The two cooperative games introduced in Chapter Two, above, raise two interesting questions.

1. How should we proceed to define the (usual) core of a game $\Gamma(\underline{e})$?

2. Assuming that the (usual) core of a game $\Gamma(\underline{e})$ has been defined, should we use the core of the game $\Gamma(\underline{e})$ or the core of the game $\tilde{\Gamma}(\underline{e})$ as the "core of an economy $\underline{e}$?" What about if more than a single core-concept can be defined for each game?

The first question, above, is not a trivial one for the following reason. In a cooperative game $\Gamma(\underline{e})$, we can either consider single coalition actions $\underline{a}_{\tilde{c}}$ as strategies for non-empty coalitions, which (see Section 2.6) will not necessarily exhaust the strategic possibilities of its members, or we can use vectors of coalition actions $\underline{a}(\tilde{c})$ as conditional strategies, which may require the consent of players in $(N-\tilde{c})$. Either approach will not necessarily indicate the "effectiveness" of a coalition for an alternative in the set $Z(\Gamma(\underline{e}))$. It also implies that it is possible to use the two approaches for defining two different core-concepts.

The general problem for defining core-concepts for a game $\Gamma(\underline{e})$ is dealt with in Section 3.2, where a general notion of dominance is introduced. In turn, this general notion of dominance induces an abstract domination relationship and thus an abstract core-concept, from which we can obtain a variety of concrete domination relationships and core-concepts by introducing various assumptions concerning the effectiveness of coalitions.

The first concrete notion of dominance and the first concrete core-concept for a game $\Gamma(\underline{e})$, the core* of that game, are introduced in Section 3.3, below. In essence, this notion of dominance is based on the idea that alternatives in the set $Z(\Gamma(\underline{e}))$ can be treated as "proposals" and "counter proposals" which can be used by non-empty coalitions, under certain conditions. In fact, a coalition need only consider the potential of its actions to lead to some alternative in order to use that alternative as a proposal, but not necessarily whether other players will be willing to go along with the corresponding to such alternative actions. Of

course those other players could refuse to go along with the proposal of that coalition if they can make a counter proposal of their own.

The second concrete notion of dominance and the second concrete core-concept for a game $\Gamma(\underline{e})$, the α -core of that game, corresponding to the "usual core," are introduced in Section 3.4, below.

The above notions of dominance and core-concepts applied to a game $\tilde{\Gamma}(\underline{e})$ (Section 3.5) could be compared directly with each other with some interesting implications. The comparison is made in Section 3.6, and the implications are discussed in Section 3.7.

One implication from such comparison is that in the absence of certain externalities in the economy, and for a cooperative game $\tilde{\Gamma}(\underline{e})$ the core* and α -core will be identical to each other. On the other hand, and as we prove (Theorem 3.3), an economy-wide externality, such as the one that will be present from the production of a pure public good, may result in the α -core of a game $\tilde{\Gamma}(\underline{e})$ to be very large while the core* of that game is empty.

This last result indicates that the answer to the second of the two questions raised at the beginning of this introduction is not so clear. From the last chapter we know that if we use the cooperative game $\tilde{\Gamma}(\underline{e})$ and not the cooperative game $\Gamma(\underline{e})$, then we may not be able to capture certain situations of conflict of interest that could exist in an economy. On the other hand, and because the provision of public goods may involve a process of public debate with proposals and counter proposals, the core* may be the more appropriate core-concept to use in such situations. It will seem then, that the answer to the second question above is not unique. However, the above discussion will indicate that

there may not be such thing as the "core of an economy" but, a core-concept relative to a cooperative game that the set of individuals in an economy might be playing.

3.2 Core-Concepts for a Game $\Gamma(\underline{e})$

Given an economic environment $\underline{e}$, let us assume that the set of individuals N play the cooperative game $\Gamma(\underline{e})$. In accordance with Chapter Two, above, this assumption implies that the set of individuals N could go ahead and make binding agreements which are equivalent to the selection of an alternative, say $(\bar{a}; \bar{y})$, from the set of alternatives of the game $\Gamma(\underline{e})$, $Z(\Gamma(\underline{e}))$, where, as may be recalled,

$$(3.1) \quad Z(\Gamma(\underline{e})) = \{(\underline{a}; \underline{y}) : \underline{a} \in \hat{A}^*, \underline{y} \in \psi(\underline{a})\}.$$

Our concern in this section is with the description of a general rule (criterion) that will determine whether the binding agreements inherent in an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ will be "stable," once such alternative has been selected by the set of individuals N .

We describe this rule through the following process. We start out from an arbitrary alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$. (We can think of this alternative as a proposal made by some referee that the set of individuals N must take under consideration.) Faced with the alternative $(\bar{a}; \bar{y})$, each non-empty group of individuals $\tilde{c} \subseteq N$, must follow one of the two Alternative Courses of Action (ACA), described below.

ACA.1: Accept the alternative $(\bar{a}; \bar{y})$. For a group of individuals $\tilde{c} \subseteq N$ to accept the alternative $(\bar{a}; \bar{y})$ means two things (a) its members must go ahead and try to make the binding agreements necessary for the

relevant (i.e., involving members of $\tilde{c}$) coalition actions in $\rho(\bar{a})$; and
 (b) each member of $\tilde{c}$ must face the prospect that $\bar{y}$ will be a final allocation if the remaining individuals (i.e., the individuals in the set $(N-\tilde{c})$ also accept the alternative $(\bar{a}; \bar{y})$.

ACA.2: Reject the alternative $(\bar{a}; \bar{y})$ and initiate the following two-step process. Step 1: They (the members of a non-empty group of individuals $\tilde{c}$ that have rejected the alternative $(\bar{a}; \bar{y})$) agree that, as a coalition, they will carry on at least one activity $r \in J$ by undertaking a coalition action $\tilde{a}_r^{\tilde{c}} \in \hat{A}_r^{\tilde{c}}$ such that $\tilde{a}_r^{\tilde{c}} \notin \rho(\bar{a})$. Step 2: Given the agreements about coalition actions reached among the players of $\tilde{c}$ in Step 1, and given that these agreements do not exhaust every member's resources, each individual $i \in \tilde{c}$ pursues to make binding agreements with others as far as his remaining resources permit, and as far as those activities with respect to which $\tilde{c}$ has not undertaken a coalition action are concerned.

A general rule for stability then, of alternatives in the set $Z(\Gamma(\underline{e}))$, could be the following. An alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ is stable if, whenever the set of individuals N are faced with this alternative, there does not exist a non-empty group of individuals $\tilde{c} \subseteq N$ that will have the incentive to follow ACA.2, the second of the two alternative courses of action described above, relative to $(\bar{a}; \bar{y})$. Alternatively, we could restate this rule as follows. An alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ is stable if, whenever the set of individuals N are faced with this alternative, they decide to form the coalition structures relevant to $\bar{a}$, and each coalition $\tilde{c} \in C(\underline{a})$ follows ACA.1.

The important question that we must answer here is the following. Under what conditions should we expect a non-empty group of individuals $\tilde{c} \subseteq N$ to follow ACA.2, the second of the two alternative courses of action described above, relative to an alternative $(\bar{a}; \bar{y})$? Apparently, this question is important because its answer will determine whether we can consider an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ stable.

To answer this question let us observe the following. Given an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, if any group of individuals $\tilde{c} \subseteq N$ chooses ACA.2, relative to this alternative, then the assumption that the set of individuals N in an economy $\underline{e}$ play the cooperative game $\Gamma(\underline{e})$ will imply that the whole process (initiated by the members of $\tilde{c}$ through ACA.2), potentially, must lead to some alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ which is different from the alternative $(\bar{a}; \bar{y})$. Therefore, a reasonable answer to the above question will be: "whenever the group of individuals $\tilde{c}$ can be assured that by following ACA.2, relative to $(\bar{a}; \bar{y})$, their actions will lead to an alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ such that

$$(3.2) \quad \tilde{y} \succeq_i \bar{y}, \forall i \in \tilde{c}, \text{ and } \tilde{y} >_s \bar{y} \text{ for some } s \in \tilde{c}." (*)$$

From now on, and whenever for any two alternatives $(\bar{a}; \bar{y})$ and $(\tilde{a}; \tilde{y})$ the respective final outcomes $\tilde{y}$ and $\bar{y}$ satisfy condition

(*) An alternative approach to (3.2) will be to postulate a strong preference such as $\tilde{y} >_1 \bar{y}, \forall i \in \tilde{c}$. For example, this will provide an incentive for every player $i \in \tilde{c}$ to follow ACA.2 relative to $(\bar{a}; \bar{y})$. Although both approaches have been used in the literature, here we have elected the approach of using a weak preference in order to indicate that the existence of Pareto-improving allocations may be a source of instability.

(3.2) with respect to some non-empty group of individuals $\tilde{c} \subseteq N$, we will say that the final outcome $\tilde{y}$ is preferred to the final outcome $\bar{y}$ by coalition $\tilde{c}$.

Now, and although an expression like: "a group of individuals $\tilde{c} \subseteq N$ can be assured that their actions will lead to an alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ " may be open to a number of interpretations, our answer to the question raised above leads to a general notion of "dominance," i.e., a general rule that will determine which ones of the alternatives in the set $Z(\Gamma(\underline{e}))$ we could consider unstable. In turn, this general notion of dominance induces a binary relationship over the set of final outcomes $\hat{Y}^*$ which we proceed to define.

Definition 3.1: Given an n-person cooperative game $\Gamma(\underline{e})$, let $(\tilde{a}; \tilde{y}), (\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$. We say that the final outcome $\tilde{y}$ dominates the final outcome $\bar{y}$, and we write: $\tilde{y} \text{ dom } \bar{y}$, if and only if there exists a coalition $\tilde{c}, \tilde{c} \in C_N, \tilde{c} \neq \emptyset$, whose members, by following ACA.2 relative to $(\bar{a}; \bar{y})$, can be assured of a non-empty set of alternatives $B, B \subseteq Z(\Gamma(\underline{e}))$, such that: (a) $((\underline{a}; \underline{y}) \in B) \Rightarrow (\underline{y} \text{ is preferred to } \bar{y} \text{ by coalition } \tilde{c})$, and (b) $(\tilde{a}; \tilde{y}) \in B$.

For any final outcome $\bar{y}, \bar{y} \in \hat{Y}^*$, let us define the set $B_{\Gamma(\underline{e})}(\bar{y})$ by.

$$(3.3) \quad B_{\Gamma(\underline{e})}(\bar{y}) \equiv \{\tilde{y} \in \hat{Y}^*: \tilde{y} \text{ dom } \bar{y}\}.$$

I.e., the set $B_{\Gamma(\underline{e})}(\bar{y})$ consists of all final outcomes in the set $\hat{Y}^*$ that dominate the final outcome $\bar{y}$.

On the basis of the binary relationship dom we can define a core-concept for a game $\Gamma(\underline{e})$ as follows.

Definition 3.2: The core of an n-person cooperative game $\Gamma(\underline{e})$, consists of the set of final outcomes $K(\Gamma(\underline{e}))$, where. $K(\Gamma(\underline{e})) \equiv \{\bar{y} \in \hat{Y}^* :$

$$B_{\Gamma(\underline{e})}(\bar{y}) = \emptyset\}.$$

Returning to the general rule for stability of alternatives in the set $Z(\Gamma(\underline{e}))$, that rule could be restated here, using Definition 3.2, as follows. An alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ is stable if the corresponding final outcome $\bar{y}$ belongs to the core of the game $\Gamma(\underline{e})$, i.e., if $\bar{y} \in K(\Gamma(\underline{e}))$.

Let us emphasize here that how we interpret the binary relationship dom, introduced in Definition 3.1, will depend on how we interpret a statement like: "the members of a non-empty coalition can be assured that their actions will lead to an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$." In other words, only after we specify what exactly we mean by such statement we can obtain a well defined binary relationship dom over the set $\hat{Y}^*$. This means that: (a) the core of a game $\Gamma(\underline{e})$, in the form given in Definition 3.2, should be understood only as an abstract core-concept for that game, and (b) concrete notions of dominance, and hence concrete core-concepts for a game $\Gamma(\underline{e})$, will be obtained only after we specify the concrete conditions under which "the members of a non-empty coalition can be assured that their actions will lead to an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$." For that matter, it may be possible that under different conditions we will obtain different concrete notions of dominance, and hence different concrete core-concepts for a game $\Gamma(\underline{e})$. Two such core-concepts are introduced in the following two sections.

3.3 The Core* of a Game $\Gamma(e)$

The concrete notion of dominance that we consider in this section is quite simple. It is obtained from the general notion of dominance introduced in the preceding section by assuming the following. The members of a non-empty coalition $\tilde{c} \in C_N$ can be assured of any alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(e))$ as long as such alternative implies that coalition $\tilde{c}$ could have followed ACA.2 relative to some other alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(e))$.

This concrete notion of dominance then induces a binary relationship over the set of final outcomes $\hat{Y}^*$ which we proceed to define.

Definition 3.3: Given an n-person cooperative game $\Gamma(e)$, let $(\tilde{a}; \tilde{y}), (\bar{a}; \bar{y}) \in Z(\Gamma(e))$. We say that the final outcome $\tilde{y}$ dominates* the final outcome $\bar{y}$, and we write $\tilde{y} \text{ dom}^* \bar{y}$, if and only if there exists a non-empty coalition $\tilde{c} \in C_N$ such that: (a) $\tilde{c} \in C(\tilde{a})$, (b) $\nexists j, j \in J$, and $\nexists \tilde{a}_j^c; \tilde{a}_j^c \in \rho(\tilde{a}), \quad \exists: \tilde{a}_j^c \notin \rho(\tilde{a}),$ and (c) $\tilde{y} \geq_i \bar{y}, \forall i \in \tilde{c}$, and $\tilde{y} >_s \bar{y}$ for some $s \in \tilde{c}$.

In the above definition, conditions (a) and (b) are equivalent to saying that the alternative $(\tilde{a}; \tilde{y})$ could have been the result of having coalition $\tilde{c}$ follow ACA.2 relative to $(\bar{a}; \bar{y})$. This, compared with Definition 3.1 will imply that the set of alternatives B , used in that definition, need only contain the element $(\tilde{a}; \tilde{y})$. In turn, this is equivalent to the assumption stated at the beginning of this section.

To get an intuitive feeling about the above concrete notion of dominance we can think as follows.

In the bargaining process that takes place in a game $\Gamma(e)$, alternatives in the set $Z(\Gamma(e))$ represent nothing more than proposals that the set of players N could consider for adoption. Therefore, the

members of a non-empty coalition $\tilde{c} \in C_N$ faced with some specific alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ that has been proposed for adoption, could either accept it, or reject it in favor of another alternative, say the alternative $(\tilde{a}; \tilde{y})$. But then, we could say that the rejection of the alternative $(\bar{a}; \bar{y})$ in favor of the alternative $(\tilde{a}; \tilde{y})$ by the members of some coalition $\tilde{c}$, $\tilde{c} \in C_N$, is equivalent to the following. The members of coalition $\tilde{c}$ use the alternative $(\tilde{a}; \tilde{y})$ as a counter proposal against the alternative $(\bar{a}; \bar{y})$.

This line of reasoning suggests that we may get a better intuitive feeling about the concrete notion of dominance that induces the binary relationship dom^* by treating it in an axiomatic way. To see how this can be done let us consider the following two axioms, stated relative to a cooperative game $\Gamma(\underline{e})$.

Axiom 1 (Existence of a Threat): Let $(\bar{a}; \bar{y})$ be any alternative in the set $Z(\Gamma(\underline{e}))$. We say that there exists a threat against the alternative $(\bar{a}; \bar{y})$ if and only if there exists another alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ and there exists a non-empty coalition $\tilde{c} \in C_N$ such that (a) $\tilde{c} \in C(\tilde{a})$ (b) $\exists j, j \in J$, and $\exists \tilde{a}_j^{\tilde{c}}, \tilde{a}_j^{\tilde{c}} \in \rho(\tilde{a}), \exists : \tilde{a}_j^{\tilde{c}} \notin \rho(\bar{a})$, and (b) $\tilde{y} \geq_1 \bar{y}$, $\forall i \in \tilde{c}$, and $\tilde{y} >_s \bar{y}$ for some $s \in \tilde{c}$.

Axiom 2 (Equal Treatment of Relevant Alternatives): An alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ that, in the sense of Axiom 1, above, constitutes a threat against an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ should have as much of a chance to be considered for adoption by the set of players N as the alternative $(\bar{a}; \bar{y})$ does, whenever at some stage of the bargaining process in a game $\Gamma(\underline{e})$ the alternative $(\bar{a}; \bar{y})$ comes up for adoption.

Let us suppose now that given any two alternatives, $(\bar{a}; \bar{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, what holds true is that: $\bar{y} \underline{\text{dom}}^* \bar{y}$. But, in a sense, this could be equivalent to saying that the alternative $(\bar{a}; \bar{y})$ has been rendered unstable on the basis of Axioms 1 and 2, above. That is, the existence of a threat against the alternative $(\bar{a}; \bar{y})$ which can be used as such on the basis of Axiom 2 is sufficient to render the alternative $(\bar{a}; \bar{y})$ unstable. Therefore, and using Axioms 1 and 2, above, we can prove the following lemma.

Lemma 3.1: Given a cooperative game $\Gamma(\underline{e})$, suppose that an alternative $(\bar{a}; \bar{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, satisfies Axiom 1. Then, (Axiom 2) $\Rightarrow$ $(\exists (\bar{a}; \bar{y}), (\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e})), \exists: \bar{y} \underline{\text{dom}}^* \bar{y})$.

Proof: Let us observe that conditions (a) - (c) in Axiom 1 are identical to conditions (a) - (c) of Definition 3.3. This may lead one to believe that the satisfaction of Axiom 1 by the alternative $(\bar{a}; \bar{y})$ is sufficient to prove the Lemma. However, this is incorrect because, the existence of a threat against an alternative $(\bar{a}; \bar{y})$ does not guarantee that it can be used. For that matter, any concrete notion of dominance derived from the general notion of dominance that induces the (abstract) binary relationship dom in Definition 3.1, requires the existence of a threat in the sense of Axiom 1. Therefore, the satisfaction of Axiom 1 by the alternative $(\bar{a}; \bar{y})$, in the lemma, is only a necessary condition for a threat $(\bar{a}; \bar{y})$ to be such that $\bar{y} \underline{\text{dom}}^* \bar{y}$. To make the existence of a threat a sufficient condition as well we need an axiom like Axiom 2, which will specify when, and what threats can be used.

From now on, the proof of the lemma follows, because the coalition that possess the threat against the alternative $(\bar{a}; \bar{y})$, according to Axiom 1, can go ahead and use it, demanding "Equal Treatment of Relevant Alternatives," i.e., on the basis of Axiom 2.

q.e.d.

Two implications that follow from Lemma 3.1, above, about the concrete notion of dominance that induces the binary relationship $\underline{\text{dom}}^*$ are the following.

1. The sense in which a coalition $\tilde{c} \in C_N$ can be assured of an alternative $(\tilde{a}; \tilde{y})$, if its members follow ACA.2 relative to another alternative $(\bar{a}; \bar{y})$, can be determined by Axiom 2. This, could be interpreted as saying the following. All that a coalition can be assured of, if its members follow ACA.2 relative to an alternative $(\bar{a}; \bar{y})$, is that, under certain conditions, another alternative will be considered for adoption, whenever the alternative $(\bar{a}; \bar{y})$ comes up for adoption by the set of players N .

2. The members of a non-empty coalition, who happen to disagree with the replacement of an alternative $(\bar{a}; \bar{y})$ with $(\tilde{a}; \tilde{y})$ as the alternative to be taken under consideration, cannot refuse to go along with the relevant coalition actions in $\rho(\tilde{a})$, except for the case that they possess a threat against the alternative $(\tilde{a}; \tilde{y})$, in the sense of Axiom 1, which they can also use on the basis of Axiom 2.

The second of the two implications stated above, is particularly important for understanding the concrete notion of dominance considered in this section. For example, and by using Axioms 1 and 2 to interpret this concrete notion of dominance, the above implication will be equivalent

to saying that no alternative in the set $Z(\Gamma(\underline{e}))$ could be considered as the status quo if there exists a threat against it. But then, and in this sense, the interpretation of elements in the set $Z(\Gamma(\underline{e}))$ as proposals becomes even more plausible.

For any final outcome $\bar{y}$, $\bar{y} \in \hat{Y}^*$ let us denote by $B_{\Gamma(\underline{e})}^*(\bar{y})$ the set of final outcomes in $\hat{Y}^*$ that dominate^{*} the final outcome $\bar{y}$, i.e., let

$$(3.4) \quad B_{\Gamma(\underline{e})}^*(\bar{y}) \equiv \{\tilde{y} \in \hat{Y}^* : \tilde{y} \text{ dom}^* \bar{y}\}.$$

The relevant core-concept obtained from the binary relationship dom^{*} is given in the following definition.

Definition 3.4: The core^{*}, of an n-person cooperative game $\Gamma(\underline{e})$, consists of the set of final outcomes $K^*(\Gamma(\underline{e}))$, where: $K^*(\Gamma(\underline{e})) \equiv \{\bar{y} \in \hat{Y}^* : B_{\Gamma(\underline{e})}^*(\bar{y}) = \emptyset\}$.

In words, Definition 3.4 says the following. The core^{*} of a game $\Gamma(\underline{e})$ consists of all final allocations in the set $\hat{Y}^*$ which are not dominated^{*} by any other final allocations in the same set. Therefore, and using the interpretation of the concrete notion of dominance that induces the binary relationship dom^{*} that will follow from Lemma 3.1, to say that a final allocation $\bar{y}$ belongs to the core^{*} of a game $\Gamma(\underline{e})$ could be interpreted as saying the following. Given an economic environment $\underline{e}$: (a) the final allocation $\bar{y}$ is obtainable from a feasible consistent joint action $\bar{a} \in \hat{A}^*$ such that the pair $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$; and (b) no coalition $\bar{c} \in C_N$, $\bar{c} \neq \emptyset$, possesses a threat (in the sense of Axiom 1) that could be used against $(\bar{a}; \bar{y})$ on the basis of the Axiom of Equal Treatment of Relevant Alternatives (Axiom 2) whenever, at some stage of

the bargaining process in the game $\Gamma(\underline{e})$, the alternative $(\bar{a}; \bar{y})$ comes up as a proposal for adoption.

In concluding this section we should point out the following. Equal Treatment of Relevant Alternatives (Axiom 2), as a rule for using threats against alternatives in the set $Z(\Gamma(\underline{e}))$, induces a notion of stability that many would consider quite strong. However, the interpretation of alternatives in the set $Z(\Gamma(\underline{e}))$ as proposals and not as actual positions from which the players must depart, make the use of Axiom 2 quite plausible. For example, proposals are just that, proposals. There is no reason why a proposal which is preferred by some coalition should be treated differently than another proposal which is preferred by another coalition. But then, the core^{*} of a game $\Gamma(\underline{e})$ could prove to be a very useful core-concept for economic environments $\underline{e}$ where alternatives in the set $Z(\Gamma(\underline{e}))$ can be interpreted as proposals. For example, economic environments $\underline{e}$ that deal with the provision of public goods will be such case.

3.4 The α -Core of a Game $\Gamma(\underline{e})$

For the concrete notion of dominance introduced in the preceding section, it was sufficient to consider single coalition actions, as strategies for the respective coalitions. In particular, and due to the Axiom of Equal Treatment of Relevant Alternatives, a non-empty coalition $\tilde{C}$, using an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, as a "counter proposal" against another alternative $(\bar{a}; \bar{y})$, the latter considered as a "proposal," did not have to be concerned whether players in $(N - \tilde{C})$ will go along with the relevant coalition actions.

In this section we deal with a concrete notion of dominance where a given coalition should be concerned with the above question before it decides to make its move. Thus, we shall consider the case where, in the terminology of Chapter Two, coalitions may have to use conditional strategies.

Let us recall, from Section 2.6, Chapter Two, that, given a cooperative game $\Gamma(\underline{e})$, and given an alternative $(\bar{a}; \bar{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, can use a conditional strategy $\tilde{a}(\tilde{c})$, if $\tilde{a}(\tilde{c})$ belongs to the consent correspondence of the game $\Gamma(\underline{e})$, i.e., if $\tilde{a}(\tilde{c}) \in \zeta(\tilde{c}; (\bar{a}; \bar{y}))$.

Given an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, let us suppose now that a non-empty coalition $\tilde{c} \in C_N$ elects to follow ACA.2, the second of the two alternative courses of action introduced in Section 3.2, above. Furthermore, let us suppose that before coalition $\tilde{c}$ decides to go ahead and use the coalition action $\tilde{a}_{\tilde{c}}^{\tilde{c}}$, indicated by Step 1, of ACA.2, it wants to make sure that the agreements pursued by its members, according to Step 2, of ACA.2, can be actually made. This is equivalent to saying that coalition $\tilde{c}$, in following ACA.2, relative to the alternative $(\bar{a}; \bar{y})$, will use a conditional strategy $\tilde{a}(\tilde{c})$, such that

$$(3.5) \quad \tilde{a}(\tilde{c}) \in \zeta(\tilde{c}; (\bar{a}; \bar{y})), \text{ and}$$

$$(3.6) \quad \exists \tilde{a}_{\tilde{c}}^{\tilde{c}}, \tilde{a}_{\tilde{c}}^{\tilde{c}} \in \tilde{a}(\tilde{c}), \ni : \tilde{a}_{\tilde{c}}^{\tilde{c}} \notin \rho(\bar{a}).$$

Note that condition (3.6) follows from Step 1, of ACA.2.

Let us observe that a conditional strategy $\tilde{a}(\tilde{c})$ which satisfies conditions (3.5) and (3.6), although it may require some actions by players in $(N-\tilde{c})$, it does not indicate what other actions those players in

$(N-\tilde{c})$ might undertake with respect to coalition actions that do not involve players from coalition $\tilde{c}$. Therefore, and by using the conditional strategy $\tilde{a}(\tilde{c})$, all that coalition $\tilde{c}$ might expect is that after every binding agreement has been completed among all players in $(N-\tilde{c})$, the result will be some joint action $\tilde{a} \in \hat{A}^*$, such that $\rho^{\tilde{c}}(\tilde{a}) = (\tilde{a}(\tilde{c}); \tilde{a}(N-\tilde{c}))$. But then, given an alternative $(\tilde{a}; \tilde{y})$ and given that some non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, uses the conditional strategy $\tilde{a}(\tilde{c})$, that satisfies condition (3.5) and (3.6), above, the set $\theta(\tilde{a}(\tilde{c}); (\tilde{a}; \tilde{y}))$, where

$$(3.7) \quad \theta(\tilde{a}(\tilde{c}); (\tilde{a}; \tilde{y})) \equiv \{\underline{a} \in \hat{A}^* : \tilde{c} \in C(\underline{a}), \\ \rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c})), \text{ and } \underline{a}(\tilde{c}) = \tilde{a}(\tilde{c})\},$$

will indicate all the strategies $\underline{a}(N-\tilde{c})$ that can be used by $(N-\tilde{c})$, the complement of coalition $\tilde{c}$, given that coalition $\tilde{c}$ uses the conditional strategy $\tilde{a}(\tilde{c})$ relative to the alternative $(\tilde{a}; \tilde{y})$, and given that, in part, players in $(N-\tilde{c})$ have agreed to such strategy as that might be required.

Given. (a) an arbitrary joint action $\tilde{a} \in \theta(\tilde{a}(\tilde{c}); (\tilde{a}; \tilde{y}))$, and (b) an arbitrary final allocation $\tilde{y} \in \psi(\tilde{a})$, the alternative $(\tilde{a}; \tilde{y})$ is one of the possible alternatives that could result if coalition $\tilde{c}$, by following ACA.2 relative to $(\tilde{a}; \tilde{y})$, uses the conditional strategy $\tilde{a}(\tilde{c})$ that satisfies conditions (3.5) and (3.6). For that matter, we can go one step further and make the following statement. A non-empty coalition $\tilde{c} \in C_N$, which follows ACA.2 relative to an alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ by using the conditional strategy $\tilde{a}(\tilde{c})$ as above, can be assured of some alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ such that $\tilde{a} \in \theta(\tilde{a}(\tilde{c}); (\tilde{a}; \tilde{y}))$. However, the specific alternative that coalition $\tilde{c}$ can be assured of, may depend on the specific strategies $\underline{a}(N-\tilde{c})$, used by the players in $(N-\tilde{c})$,

after these players fulfill their obligations regarding the relevant coalition actions in $\tilde{a}(\tilde{c})$.

Under the same conditions used to define the correspondence $\theta(\cdot)$ in (3.7), let us define the set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ as follows.

$$(3.8) \quad S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y})) \equiv \{(\underline{a}; \underline{y}) \in Z(\Gamma(\underline{e})): \underline{a} \in \theta(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))\}$$

In words, a non-empty coalition $\tilde{c} \in C_N$, which follows ACA.2 relative to an alternative $(\bar{a}; \bar{y})$ by using the conditional strategy $\tilde{a}(\tilde{c})$, can be assured of some alternative in the set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$. Hence, the set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ represents all possible alternatives that could result if a non-empty coalition $\tilde{c} \in C_N$ follows ACA.2 relative to the alternative $(\bar{a}; \bar{y})$ by using the conditional strategy $\tilde{a}(\tilde{c})$.

Let us suppose now that for some alternative $(\bar{a}; \bar{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, the following hold true (a) there exists a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, (b) there exists a conditional strategy for that coalition, say the strategy $\tilde{a}(\tilde{c})$, which, relative to the alternative $(\bar{a}; \bar{y})$, satisfies conditions (3.5) and (3.6), above, and (c) the corresponding set of alternatives $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ is such that

$$(3.9) \quad ((\underline{a}; \underline{y}) \in S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))) \Rightarrow (\underline{y} \succeq_i \bar{y}, \forall i \in \tilde{c}, \\ \text{and } \underline{y} >_s \bar{y}, \text{ for some } s \in \tilde{c}).$$

Apparently, for any alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ that satisfies this assumption, there exists a non-empty coalition $\tilde{c}$, and there exists a conditional strategy $\tilde{a}(\tilde{c})$, for that coalition, such that the following holds true. If this coalition follows ACA.2 relative to $(\bar{a}; \bar{y})$ by using the strategy $\tilde{a}(\tilde{c})$, then, and independently of the strategies $\underline{a}(N-\tilde{c})$ used

by its complement $(N-\tilde{c})$ (i.e., independently of any joint strategy $\underline{a}$, $\underline{a} \in \theta(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$, that could result from such process), it is always guaranteed that any alternative, which can result from that process, will be such that the corresponding final allocation will be preferred to the final allocation $\bar{y}$ by coalition $\tilde{c}$.

This conclusion implies that we can define a new concrete notion of dominance in a game $\Gamma(\underline{e})$ that will render an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$ unstable if it satisfies the above assumption. In turn, this will be equivalent to imposing the requirement that a non-empty coalition $\tilde{c} \in C_N$ which follows ACA.2 relative to an alternative $(\bar{a}; \bar{y})$, can be assured of an alternative $(\tilde{a}; \tilde{y})$ that satisfies condition (3.2) if and only if it can use a conditional strategy $\tilde{a}(\tilde{c})$ such that $\tilde{a}(\tilde{c}) \in \rho^{\tilde{c}}(\tilde{a})$, and conditions (3.5), (3.6), and (3.9) are satisfied relative to $(\bar{a}; \bar{y})$.

The concrete notion of dominance which can be obtained through such requirement induces a binary domination relationship over the set of final allocations $\hat{Y}^*$ of a game $\Gamma(\underline{e})$, which we proceed to define.

Definition 3.5: Given an n-person cooperative game $\Gamma(\underline{e})$, let $(\tilde{a}; \tilde{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$. We say that the final outcome $\tilde{y}$ α -dominates the final outcome $\bar{y}$, and we write $\tilde{y} \alpha\text{-dom } \bar{y}$, if and only if there exists a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, such that (a) $\tilde{c} \in C(\bar{a})$, $\rho^{\tilde{c}}(\tilde{a}) = (\tilde{a}(\tilde{c}); \tilde{a}(N-\tilde{c}))$, and $\tilde{a}(\tilde{c})$ satisfies conditions (3.5) and (3.6), (b) $(\tilde{a}; \tilde{y}) \in S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$, and (c) the set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ satisfies condition (3.9).

In words, we could describe the binary relationship "alpha-domination," defined over the set of final outcomes (allocation) $\hat{Y}^*$, as follows. Given any two alternatives $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$ in the set $Z(\Gamma(\underline{e}))$, we say that the final outcome $\tilde{y}$ alpha-dominates the final outcome

$\bar{y}$ if and only if there exists a non-empty $\tilde{a}$ -admissible coalition $\tilde{c} \in C(\tilde{a})$, such that, (a) the corresponding, to the reordering $\rho^{\tilde{c}}(\tilde{a})$ of $\tilde{a}$, vector of coalition actions $\tilde{a}(\tilde{c})$ can be used as a conditional strategy by coalition $\tilde{c}$, in the sense of (3.5) and (3.6), relative to the alternative $(\bar{a}; \bar{y})$, (b) for each strategy $\underline{a}(N-\tilde{c})$ used by $(N-\tilde{c})$ under the rules of the game $\Gamma(\underline{e})$, the resulting joint strategies cannot yield final outcomes which are not preferred to the final outcome $\bar{y}$ by coalition $\tilde{c}$, and (c) $\bar{y}$ is among such outcomes. (*)

It is of interest to note here that if a coalition $\tilde{c} \in C_N$ uses a conditional strategy $\tilde{a}(\tilde{c})$ that satisfies conditions (3.5) and (3.6) relative to an alternative $(\bar{a}; \bar{y})$, and if the relevant set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ satisfies condition (3.9) then, coalition $\tilde{c}$ can always count on a final outcome which it will prefer to the final outcome $\bar{y}$, independently of the strategies used by its complement $(N-\tilde{c})$. But this is equivalent to saying that coalition $\tilde{c}$ is effective for a set of outcomes which it prefers to $\bar{y}$. Since this notion of effectiveness of a coalition in a game $\Gamma(\underline{e})$ is independent (albeit a restrictive sense of independence) of the strategies of its complement, it corresponds to the notion of α -effectiveness introduced by Aumann and Peleg [5]. It is for this reason that we call the domination relationship in Definition 3.5, "alpha-domination." Apparently, the set of outcomes for which a coalition $\tilde{c} \in C_N$ is α -effective

(*) Under the rules of the game $\Gamma(\underline{e})$, the satisfaction of conditions (3.5) and (3.6) by $\tilde{a}(\tilde{c})$ implies that the complement $(N-\tilde{c})$ of coalition $\tilde{c}$ is restricted to choose strategies $\underline{a}(N-\tilde{c})$ such that for some $\underline{a} \in \hat{A}^*$, $\rho^{\tilde{c}}(\underline{a}) = (\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c}))$ and $\underline{a}(\tilde{c}) = \tilde{a}(\tilde{c})$.

if it uses a strategy $\tilde{a}(\tilde{c})$, which satisfies conditions (3.5) and (3.6), relative to an alternative $(\bar{a}; \bar{y})$, are those outcomes that correspond to the alternatives in the set $S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$.

For any final outcome (allocation) $\bar{y} \in \hat{Y}^*$, let us denote by $B_{\Gamma(\underline{e})}^{\alpha}(\bar{y})$ the set of final outcomes in $\hat{Y}^*$ that α -dominate $\bar{y}$, i.e., let

$$(3.10) \quad B_{\Gamma(\underline{e})}^{\alpha}(\bar{y}) \equiv \{\bar{y} \in \hat{Y}^* : \bar{y} \text{ } \underline{\alpha\text{-dom}} \bar{y}\}.$$

The relevant core-concept, obtained from the binary relationship α -dom, is described in the following definition.

Definition 3.6: The α -core, of an n-person cooperative game $\Gamma(\underline{e})$, consists of the set of final outcomes, $K^{\alpha}(\Gamma(\underline{e}))$, where $K^{\alpha}(\Gamma(\underline{e})) \equiv \{\bar{y} \in \hat{Y}^* :$

$$B_{\Gamma(\underline{e})}^{\alpha}(\bar{y}) = \emptyset\}.$$

In words, the α -core of a game $\Gamma(\underline{e})$ consists of all final outcomes (allocations) in the set $\hat{Y}^*$ which are not α -dominated by any other allocations in the same set. Therefore, to say that a final outcome $\bar{y} \in \hat{Y}^*$ belongs to the α -core of a game $\Gamma(\underline{e})$ could be interpreted as saying the following. Given an economic environment $\underline{e}$: (a) the final outcome (allocation) $\bar{y}$ is obtainable from a feasible consistent joint action $\bar{a} \in \hat{A}^*$ (i.e., there exists $\bar{a} \in \hat{A}^*$ such that $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$); and (b) no coalition $\tilde{c} \in C_N$, $\tilde{c} \neq \emptyset$, can use a conditional strategy $\tilde{a}(\tilde{c})$ that (a) satisfies conditions (3.5) and (3.6) relative to $(\bar{a}; \bar{y})$, and (b) will make it α -effective set of final outcomes which it prefers to $\bar{y}$.

Now, the core-concept of a cooperative game which is defined on the basis of α -effectiveness represents the "usual core" of that game. Therefore, and if there exists a core-concept that we could refer to, as the "usual core of a game $\Gamma(\underline{e})$," that core-concept must be the α -core of $\Gamma(\underline{e})$.

3.5 Core-Concepts for a Game $\tilde{\Gamma}(\underline{e})$

Given an economic environment $\underline{e}$, the n -person cooperative game $\tilde{\Gamma}(\underline{e})$, introduced in Section 2.7, of Chapter Two, above, differs from the cooperative game $\Gamma(\underline{e})$ only on the set of joint strategies that the set of players are allowed to use in each game. In particular, the game $\tilde{\Gamma}(\underline{e})$ has been defined in (2.17) by postulating that the set of individuals N , in an economy $\underline{e}$, cannot use joint actions that belong to the set $(\hat{A}^* - \tilde{A})$.

Let,

$$(3.11) \quad Z(\tilde{\Gamma}(\underline{e})) \equiv \{(\underline{a}; \underline{y}) : \underline{a} \in \tilde{A}, \underline{y} \in \psi(\underline{a})\}.$$

Then, the set $Z(\tilde{\Gamma}(\underline{e}))$ represents the set of alternatives for the game $\tilde{\Gamma}(\underline{e})$. Furthermore, and because $\tilde{A} \subseteq \hat{A}^*$, $Z(\tilde{\Gamma}(\underline{e}))$ is a restriction of alternatives in the set $Z(\Gamma(\underline{e}))$, i.e., $Z(\tilde{\Gamma}(\underline{e})) \subseteq Z(\Gamma(\underline{e}))$. This could be interpreted as saying that a change in the rules of the game $\Gamma(\underline{e})$ so that, in effect, the set of individuals N in an economy $\underline{e}$ will end up playing the cooperative game $\tilde{\Gamma}(\underline{e})$, in a sense, it is equivalent to imposing the restriction that the only alternatives in $Z(\Gamma(\underline{e}))$ which should be relevant to our analysis are those that belong to the set $Z(\tilde{\Gamma}(\underline{e}))$. But then, Definitions 3.1 - 3.6, in the preceding three sections, which have been stated relative to alternatives in the set $Z(\Gamma(\underline{e}))$ could be restated in terms of alternatives in the set $Z(\tilde{\Gamma}(\underline{e}))$ as well.

The above conclusion implies that the three notions of dominance introduced in the preceding three sections relative to a game $\Gamma(\underline{e})$, are such that their validity holds for a game $\tilde{\Gamma}(\underline{e})$, as well. Furthermore, if $\tilde{Y}$ denotes the set of final outcomes (allocations) which, in an economy $\underline{e}$, could result from joint actions in the set $\tilde{A}$, i.e., if

$$(3.12) \quad \tilde{Y} \equiv \bigcup_{\underline{a} \in \tilde{A}} \psi(\underline{a})$$

then, and relative to a game $\tilde{\Gamma}(\underline{e})$, the three binary domination relationships, dom, dom^{*}, and α -dom (see Definitions 3.1, 3.3, and 3.5, respectively) which can be induced from the above three notions of dominance, will constitute restrictions of the original ones (i.e., the ones defined relative to a game $\Gamma(\underline{e})$) over the set $\tilde{Y}$. Likewise, and through such restriction of the binary relationships dom, dom^{*}, and α -dom over the set $\tilde{Y}$, we can obtain the respective core, core^{*}, and α -core of a game $\tilde{\Gamma}(\underline{e})$, relative to the set of final outcomes $\tilde{Y}$.

As may be recalled from Chapter Two, Section 2.7, in a game $\tilde{\Gamma}(\underline{e})$, vectors of coalition actions $\underline{a}(\tilde{c})$, any $\tilde{c} \in C_N$, are of the form

$$(3.13) \quad \underline{a}(\tilde{c}) = (\underline{a}_j^{\tilde{c}})_{j \in J},$$

and they constitute independent strategies for the respective coalitions. This implies that the consent correspondence of a game $\tilde{\Gamma}(\underline{e})$, which we shall denote by $\tilde{\zeta}(\cdot)$, takes the specific form

$$(3.14) \quad \tilde{\zeta}(\tilde{c}; (\bar{a}; \bar{y})) \equiv \{\tilde{a}(\tilde{c}): \exists \tilde{a}, \tilde{a} \in \tilde{A}, \tilde{c} \in C(\underline{a}), \\ \text{and } \rho^{\tilde{c}}(\tilde{a}) = (\tilde{a}(\tilde{c}); \tilde{a}(N-\tilde{c}))\}.$$

In other words, the set $\tilde{\zeta}(\tilde{c}; (\bar{a}; \bar{y}))$ contains every vector of coalition actions $\tilde{a}(\tilde{c})$, compatible with some joint action $\tilde{a} \in \tilde{A}$ that admits coalition $\tilde{c}$. Thus, $\tilde{\zeta}(\tilde{c}; (\bar{a}; \bar{y}))$ is independent of the specific alternative $(\bar{a}; \bar{y})$. This, should be the case for a game $\tilde{\Gamma}(\underline{e})$ because the consent of players in $(N-\tilde{c})$ is completely unnecessary for coalition actions in $\tilde{a}(\tilde{c})$.

For each non-empty coalition $\tilde{c} \in C_N$, for each alternative $(\bar{a}; \bar{y}) \in Z(\tilde{\Gamma}(\underline{e}))$, and for each strategy $\tilde{a}(\tilde{c}) \in \tilde{\zeta}(\tilde{c}; (\bar{a}; \bar{y}))$, let,

$$(3.15) \quad \begin{aligned} \tilde{\theta}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y})) &\equiv \{\underline{a} \in \tilde{A}: \tilde{c} \in C(\underline{a}), \\ \rho^{\tilde{c}}(\underline{a}) &= (\underline{a}(\tilde{c}); \underline{a}(N-\tilde{c})), \text{ and } \underline{a}(\tilde{c}) = \tilde{a}(\tilde{c})\}, \text{ and} \end{aligned}$$

$$(3.16) \quad \tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y})) \equiv \{(\underline{a}; \underline{y}) \in Z(\tilde{\Gamma}(\underline{e})): \underline{a} \in \tilde{\theta}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))\}.$$

Apparently, the interpretation of the sets $\tilde{\theta}(\cdot)$ and $\tilde{S}(\cdot)$, above, is exactly the same as that of the corresponding sets $\theta(\cdot)$ and $S(\cdot)$ for a game $\Gamma(\underline{e})$, with only one difference. For any joint action $\underline{a} \in \tilde{A}$ such that $\tilde{c} \in C(\underline{a})$, some $\tilde{c} \in C_N$, the two vectors of coalition actions in the reordering $\rho^{\tilde{c}}(\underline{a})$, $\underline{a}(\tilde{c})$, and $\underline{a}(N-\tilde{c})$ contain coalition actions that involve only players in the corresponding sets $\tilde{c}$ and $(N-\tilde{c})$, respectively. For that matter, the properties of joint actions in the set $\tilde{A}$ imply that if τ is the corresponding, common coalition structure to some $\underline{a} \in \tilde{A}$, such that $\tilde{c} \in C(\underline{a})$, then, $\underline{a}(N-\tilde{c}) = ((\underline{a}_j^c)_{j \in J}^c)_{c \in (\tau - \tilde{c})}$.

Let us observe now that by using the expressions in (3.11) - (3.16), above, we could restate Definitions 3.1 - 3.6, in the preceding three sections, relative to a cooperative game $\tilde{\Gamma}(\underline{e})$ by just changing the relevant notation. In particular then,

$$(3.17) \quad K(\tilde{\Gamma}(\underline{e})) \equiv \{y \in \tilde{Y}: B_{\tilde{\Gamma}(\underline{e})}^{\sim}(y) = \emptyset\},$$

$$(3.18) \quad K^*(\tilde{\Gamma}(\underline{e})) \equiv \{y \in \tilde{Y}: B_{\tilde{\Gamma}(\underline{e})}^*(y) = \emptyset\}, \text{ and}$$

$$(3.19) \quad K^\alpha(\tilde{\Gamma}(\underline{e})) \equiv \{y \in \tilde{Y}: B_{\tilde{\Gamma}(\underline{e})}^\alpha(y) = \emptyset\},$$

where $B_{\tilde{\Gamma}(\underline{e})}^{\sim}(y)$, $B_{\tilde{\Gamma}(\underline{e})}^*(y)$, and $B_{\tilde{\Gamma}(\underline{e})}^{\alpha}(y)$, will be the sets corresponding to those in (3.3), (3.4), and (3.10), respectively, defined here relative to a game $\tilde{\Gamma}(\underline{e})$.

For a given cooperative game $\tilde{\Gamma}(\underline{e})$, whether we designate a coalition action $\tilde{a}_j^{\tilde{c}}$ or a vector of coalition actions $\tilde{a}(\tilde{c})$ as a strategy for the respective coalition $\tilde{c}$, will not make any difference. In particular, and because vectors of coalition actions $\tilde{a}(\tilde{c})$ constitute independent strategies for the respective coalitions, in the sense of (3.13), and in the sense described in Chapter Two, we can say the following about the binary relationship $\underline{\text{dom}}^*$. Relative to a game $\tilde{\Gamma}(\underline{e})$, the binary relationship $\underline{\text{dom}}^*$ can be defined, exclusively, through the elements of the sets $\tilde{S}(\cdot)$ in the sense of the following theorem.

Theorem 3.1: Given a cooperative game $\tilde{\Gamma}(\underline{e})$, let $(\tilde{a}; \tilde{y})$, $(\bar{a}; \bar{y}) \in Z(\tilde{\Gamma}(\underline{e}))$.

Then, $\tilde{y} \underline{\text{dom}}^* \bar{y}$ if and only if, there exists a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, such that: (a) $\tilde{c} \in C(\tilde{a})$, $\rho^{\tilde{c}}(\tilde{a}) = (\tilde{a}(\tilde{c}); \tilde{a}(N-\tilde{c}))$, and $\tilde{a}(\tilde{c})$ satisfies condition (3.6), (b) $(\tilde{a}; \tilde{y}) \in \tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$, and (c) $\tilde{y} \geq_{\tilde{1}} \bar{y}$, $\forall i \in \tilde{c}$, and $\tilde{y} >_{\tilde{s}} \bar{y}$ for some $s \in \tilde{c}$.

Proof: Suppose that $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$ are such that $\tilde{y} \underline{\text{dom}}^* \bar{y}$. By Definition 3.3, there exists a non-empty coalition $\tilde{c}$, $\tilde{c} \in C_N$, such that (i) $\tilde{c} \in C(\tilde{a})$, (ii) $\nexists j, j \in J$, and $\nexists \tilde{a}_j^{\tilde{c}}, \tilde{a}_j^{\tilde{c}} \in \rho(\tilde{a})$, $\exists: \tilde{a}_j^{\tilde{c}} \notin \rho(\tilde{a})$, and (iii) $\tilde{y} \geq_{\tilde{1}} \bar{y}$, $\forall i \in \tilde{c}$, and $\tilde{y} >_{\tilde{s}} \bar{y}$ for some $s \in \tilde{c}$. But (i) implies that we can reorder the elements of $\tilde{a}$ so that $\rho^{\tilde{c}}(\tilde{a}) = (\tilde{a}(\tilde{c}); \tilde{a}(N-\tilde{c}))$. Furthermore, (ii) implies that the vector of coalition actions $\tilde{a}(\tilde{c})$ satisfies condition (3.6). Hence, condition (a) of the theorem will be implied by conditions (i) and (ii), above. Apparently, condition (iii), above, is identical to condition (c) of the theorem. Therefore, it remains to show that

$(\tilde{a}; \tilde{y}) \in \tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$. But this follows trivially from the definition of the sets $\tilde{S}(\cdot)$, because $\tilde{a}(\tilde{c})$, as above, constitutes a strategy for coalition $\tilde{c}$.

The proof of the converse, i.e., that, given $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$, the existence of a non-empty coalition which satisfies conditions (a) - (c) of the theorem will imply that $\tilde{y} \underline{\text{dom}}^* \bar{y}$, follows trivially from Definition 3.3 applied to a game $\tilde{\Gamma}(\underline{e})$.

q.e.d.

The importance of Theorem 3.1 lies on the fact that it is not necessarily valid for a game $\Gamma(\underline{e})$. The reason for this, is the following. Under the rules of a game $\Gamma(\underline{e})$, vectors of coalition actions $\tilde{a}(\tilde{c})$ constitute only conditional strategies for the respective coalitions. Their use may require the consent of players in $(N - \tilde{c})$. But then, we should not expect that condition (b) of Theorem 3.1 will be always satisfied in a game $\Gamma(\underline{e})$ if $\tilde{y} \underline{\text{dom}}^* \bar{y}$. In other words, given a game $\Gamma(\underline{e})$, and given two alternatives $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$ such that $\tilde{y} \underline{\text{dom}}^* \bar{y}$, the corresponding vector of coalition actions $\tilde{a}(\tilde{c})$, of the relevant coalition $\tilde{c}$, may be such that $\tilde{a}(\tilde{c}) \notin \zeta(\tilde{c}; (\bar{a}; \bar{y}))$. Hence, the alternative $(\tilde{a}; \tilde{y})$ may be such that $(\tilde{a}; \tilde{y}) \notin S(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$. However, this cannot be the case for a game $\tilde{\Gamma}(\underline{e})$ because, in such game, it is always true, as we have pointed out in defining $\tilde{\zeta}(\cdot)$, that every vector of coalition actions $\tilde{a}(\tilde{c})$ is such that $\tilde{a}(\tilde{c}) \in \tilde{\zeta}(\tilde{c}; (\bar{a}; \bar{y}))$, for any $(\bar{a}; \bar{y}) \in Z(\tilde{\Gamma}(\underline{e}))$. As a consequence, it will always be true that, in a game $\tilde{\Gamma}(\underline{e})$, the relevant alternative $(\tilde{a}; \tilde{y})$ is such that $(\tilde{a}; \tilde{y}) \in \tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$.

3.6 Comparisons Between the Core* and the α -Core of a Game

The conclusion of Theorem 3.1, in the preceding section, provides an alternative definition for the binary relationship dom^* in a game $\tilde{\Gamma}(\underline{e})$. In particular, a definition which is based on the sets $\tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ defined in (3.16). This permits a direct comparison between the binary relationships dom^* and $\alpha\text{-dom}$ in a game $\tilde{\Gamma}(\underline{e})$, because Definition 3.5, applied to a game $\tilde{\Gamma}(\underline{e})$, could also be restated in terms of the same sets $\tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$.

To make this comparison, let us observe that the only essential difference between the conclusion of Theorem 3.1 and Definition 3.5, the latter applied to a game $\tilde{\Gamma}(\underline{e})$, is on condition (c) of the respective theorem and definition. More precisely, for the binary relationship $\alpha\text{-dom}$, in a game $\tilde{\Gamma}(\underline{e})$, every element $(\underline{a}; \underline{y}) \in \tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ must be such that the final allocation $\underline{y}$ is preferred to the final allocation $\bar{y}$ by coalition $\tilde{c}$. This follows from condition (3.9). On the other hand, for the binary relationship dom^* , in a game $\tilde{\Gamma}(\underline{e})$, it is sufficient that a set $\tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ contains only one element $(\tilde{a}; \tilde{y})$ such that $\tilde{y}$ is preferred to $\bar{y}$ by the respective coalition $\tilde{c}$.

There are three implications that follow this observation.

1. For any final outcome $\bar{y} \in \bar{Y}$, such that $B_{\tilde{\Gamma}(\underline{e})}^{\alpha}(\bar{y}) \neq \emptyset$, it will always be true that $B_{\tilde{\Gamma}(\underline{e})}^{\star}(\bar{y}) \neq \emptyset$. Therefore, $K^{\star}(\tilde{\Gamma}(\underline{e})) \subseteq K^{\alpha}(\tilde{\Gamma}(\underline{e}))$, for each game $\tilde{\Gamma}(\underline{e})$.

2. It is possible that for some game $\tilde{\Gamma}(\underline{e})$, there exists a final allocation $\bar{y} \in \bar{Y}$, such that $\bar{y} \in K^{\alpha}(\tilde{\Gamma}(\underline{e}))$ but $\bar{y} \notin K^{\star}(\tilde{\Gamma}(\underline{e}))$.

In other words, it is not always true that for a given game $\tilde{\Gamma}(\underline{e})$, $K^{\alpha}(\tilde{\Gamma}(\underline{e})) \subseteq K^{\star}(\tilde{\Gamma}(\underline{e}))$.

3. There is a sufficient condition, stated in the following theorem, which, if satisfied by a cooperative game $\tilde{\Gamma}(\underline{e})$, will guarantee that $K^\alpha(\tilde{\Gamma}(\underline{e})) = K^*(\tilde{\Gamma}(\underline{e}))$.

Theorem 3.2: Given a cooperative game $\tilde{\Gamma}(\underline{e})$, let us assume that.

$((\tilde{a}; \tilde{y}), (\bar{a}; \bar{y}) \in Z(\tilde{\Gamma}(\underline{e}))$, and $(\tilde{a}; \tilde{y}), (\bar{a}; \bar{y})$ satisfy conditions (a) - (c) in the conclusion of Theorem 3.1 with respect to coalition $\tilde{c}$
 $\Rightarrow (\tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y})))$ satisfies condition (3.9)). Then, $K^*(\tilde{\Gamma}(\underline{e})) = K^\alpha(\tilde{\Gamma}(\underline{e}))$.

Proof: It follows trivially from the first implication above, and the fact that if for any pair of alternatives as above, to satisfy conditions (a) - (c) in the conclusion of Theorem 3.1 with respect to some coalition $\tilde{c}$ means that the relevant set $\tilde{S}(\tilde{a}(\tilde{c}); (\bar{a}; \bar{y}))$ satisfies condition (3.9), then $\tilde{y} \in (B_{\tilde{\Gamma}(\underline{e})}^*(\tilde{y})) \cap (B_{\tilde{\Gamma}(\underline{e})}^\alpha(\tilde{y}))$. Hence, $B_{\tilde{\Gamma}(\underline{e})}^*(\tilde{y}) \neq \emptyset$ will imply that $B_{\tilde{\Gamma}(\underline{e})}^\alpha(\tilde{y}) \neq \emptyset$, for any $\tilde{y} \in \tilde{Y}$.

q.e.d.

The three implications stated above have some economic reasons.

We state them in the next section.

3.7 Externalities and Public Goods

The reason that the core^{*} and the α -core of a game $\tilde{\Gamma}(\underline{e})$ may not be identical to each other is not just a matter of technicality. Although the two have been defined on the basis of different notions of dominance, as Theorem 3.2 in conjunction with the preceding discussion reveal, it is the nature of the particular economic environment $\underline{e}$ which we take under consideration that will determine, in each case, whether or not, for the relevant cooperative game $\tilde{\Gamma}(\underline{e})$ those two core-concepts are identical to each other.

For example, in a pure exchange economy with "selfish" or "individualistic" preferences (in the sense of Hurwicz [17]) the above core-concepts will be identical to each other. On the other hand, even with selfish or individualistic preferences, we should not expect, in general, the above core-concepts to be identical to each other if the economic environment under consideration includes a pure public good.

What this means is that certain forms of externalities may cause the α -core to be different from the core^{*} in an economic environment e where the set of individuals play the cooperative game $\tilde{\Gamma}(\underline{e})$. However, not every form of externality will cause a discrepancy between the two core-concepts. As Theorem 3.2 reveals certain forms of externalities may be present in an economic environment e and that, will not cause a discrepancy between the core^{*} and the α -core of the relevant game $\tilde{\Gamma}(\underline{e})$. Such externalities could be called coalition specific in the sense that they can affect the members of a given coalition only but they cannot extend to the rest of the economy. For example, the production of an "excludable public good" by a coalition, is a form of a coalition specific externality. However, and assuming that in the relevant economy e the set of individuals play the cooperative game $\tilde{\Gamma}(\underline{e})$, this form of externality will not cause a discrepancy between the core^{*} and the α -core of that game.

Although coalition specific externalities will not cause a discrepancy between the core^{*} and the α -core of a game $\tilde{\Gamma}(\underline{e})$, economy wide externalities may have the opposite affect. In particular, the discrepancy between the two core-concepts caused by such externalities, may be so wide as to result in one core-concept to be empty, while the other is very large.

To prove this point let us consider the following example of an economy with one private and one pure public good, which is a version of an example presented by T. Muench [24]. (*)

Example

$N = \{1, \dots, n\}$ the set of economic agents; $n \geq 2$.
 $u^i = y^i - e^{-x}$ the i -th agent's utility function, where y^i represents consumption of the private good by the i -th agent while x represents consumption of the pure public good.
 $(\bar{y}^i; \bar{x}) = (1; 0)$ the initial endowment of the private and public good, respectively of the i -th agent.
 $x = e \cdot \ell$ the production function of the public good for $\ell \geq 0$, where ℓ represents the units of the private good used in the production of x , and e is the number whose natural logarithm, denoted by: " $\ln$ ", is equal to 1, i.e., $\ln(e) = 1$.

An allocation in this economy consists of a pair $(y; x)$, where y represents the allocation of the private good in the economy, i.e., y is the n -dimensional vector $(y^i)_{i \in N}$, and x represents the amount of the public good available for consumption by any individual $i \in N$.

An allocation $(y; x)$ for the above economy is feasible if $y^i \geq 0$, $\forall i \in N$, $x \geq 0$, and $\frac{x}{e} + \sum_{i \in N} y^i = n$.

(*) We should point out that Muench has considered a continuum of economic agents while here we consider the case with a finite set of economic agents. However, this will not affect our conclusions.

It is easy to show that for this example a feasible allocation $(y^*; x^*)$ will be a Lindahl allocation if, and only if, $x^* = 1 + \ln(n)$, and $y^{*i} = 1 - \frac{\ln(ne)}{ne}$, $\forall i \in N$. The "individualized price" that each individual $i \in N$ must pay for each unit of the public good, at the Lindahl equilibrium, denoted by p^{*i} , is such that $p^{*i} = \frac{1}{ne}$ units of the private good. Hence, at the Lindahl allocation, each individual's utility function will take the value u^{*i} , where $u^{*i} = 1 - \frac{\ln(ne)}{ne} - e^{-\ln(ne)}$.

In [24], T. Muench has shown the following with respect to an economy like the one presented in the example. (a) Individually rational allocations (i.e., allocations where no individual's utility function takes a value lower than what it takes at that individual's initial endowment) are Pareto optimal if and only if the corresponding quantity of the public good is identical to that of the Lindahl allocation ([24], Lemma 4.1). (b) The core of such an economy, which according to our terminology will be the same as the α -core of the relevant game $\tilde{\Gamma}(e)$, will not be identical with the Lindahl allocation, and for that matter it will be very large.

It is apparent, from the first of the two conclusions listed above, that allocations in the set $K^\alpha(\tilde{\Gamma}(e))$, for the particular economy e described in the example, will be different from each other only in the amount of the private good consumed by each individual $i \in N$. In turn, given n , a constant amount of the private good, equal to $\frac{\ln(ne)}{e}$, is devoted for the production of the public good in each of those allocations. Therefore, elements in $K^\alpha(\tilde{\Gamma}(e))$ will differ from each other only in the way that a total amount of $(n - \frac{\ln(ne)}{e})$ units of the private good is distributed among the set of individuals N , with the Lindahl equilibrium

representing the equal distribution where, $y^{*i} = 1 - \frac{\ln(ne)}{ne}$, $\forall i \in N$.

Therefore, if we were to show that the Lindahl distribution can be dominated^{*} by a single-person coalition, then any allocation in $K^\alpha(\tilde{\Gamma}(\underline{e}))$ will be dominated^{*} by some single-person coalition, in particular by those individuals who, at such allocations, must consume a lower quantity of the private good than y^{*i} .

To show this we use Theorem 3.1. The strategy that a player $i \in N$ uses is to withdraw from the production of the public good and keep his endowment for his own consumption. This is consistent with a feasible consistent joint action where the set of individuals $N - \{i\}$ continue to produce a level of the public good which will be Pareto optimal for an economy with $(n-1)$ individuals, i.e., where the new level of the public good produced is equal to $\ln((n-1)e)$. At this level of the public good, the consumption vector of the individual that has withdrawn from the production of the public good is such that $(\tilde{y}^i, \tilde{x}) = (1, \ln(n-1)e)$, and his utility function takes the value $\tilde{u}^i$, where $\tilde{u}^i = 1 - e^{-\ln((n-1)e)}$. Therefore, the Lindahl allocation $(y^*; x^*)$ will be dominated^{*} by the allocation $(\tilde{y}; \tilde{x})$, where: $\tilde{y}^j = 1 - \frac{\ln((n-1)e)}{(n-1)e}$, $\forall j, j \in N, j \neq i$; $\tilde{y}^i = 1$; and $\tilde{x} = \ln((n-1)e)$; if we show that $\tilde{u}^i > u^{*i}$. This is equivalent to showing that

$$1 - e^{-\ln((n-1)e)} > 1 - \frac{\ln(ne)}{ne} - e^{-\ln(ne)}$$

or that

$$1 - e^{-\ln(n-1)} e^{-1} > 1 - \frac{\ln(ne)}{ne} - e^{-\ln(n)} e^{-1}$$

which holds true for any value of $n \geq 2$. (i.e., it holds for $n = 2$, because $\ln 2 = .6931$. For larger n , $\ln(n)$ will be approximately equal to $\ln(n-1)$, so it will be reduced to $0 > -\frac{\ln(ne)}{ne}$).

What we have shown, by means of an example is that the following theorem holds true.

Theorem 3.3: There exists a class of economic environments $\underline{e}$ such that $K^\alpha(\tilde{\Gamma}(\underline{e})) \neq \emptyset$ (and "very large") but $K^*(\tilde{\Gamma}(\underline{e})) = \emptyset$.

CHAPTER FOUR
INDIVIDUALLY DEFENSIBLE POSITIONS AND
THE EFFECTIVE CORE FOR n-PERSON
COOPERATIVE GAMES

4.1 Introduction

A number of solution concepts to n-person cooperative games, such as the various bargaining sets (e.g., [4], and [8]) and the kernel (e.g., [7]), have been defined on the premise that in a given game, a payoff vector can be considered "stable" if it offers security to the players, that is, if it satisfies the safety requirements of the players in that game. However, other solution concepts, such as the core, seem to indicate that the safety requirements of the players, even if satisfied for some particular payoff vector, in a given game, may not be sufficient to induce stability, hence, that additional criteria may be needed for that purpose. Furthermore, even if we were to accept the hypothesis that the safety requirements of the players, if satisfied for some payoff vector, are sufficient to induce stability, there may be different sets of criteria for that purpose. In particular, a player may not feel safe at a payoff vector if he cannot, in a well defined sense, "defend" his corresponding payoff against those concerted actions of the other players that, potentially, could affect him most adversely.

In this chapter, we consider two such notions of "defense" for the class of n-person cooperative games where side payments are allowed and utility may be transferable. Furthermore, we consider the hypothesis that certain solution concepts to such cooperative games can be defined

on the basis of both, a "security criterion" and a "maximum benefit criterion."

In [4], R. J. Aumann and M. Maschler advanced the hypothesis that, in a cooperative game, a payoff vector could be considered "stable" if, in some sense, it offers security to each player. In support of this hypothesis, a number of solution concepts, the various bargaining sets, were introduced in [4]. Payoff vectors in a bargaining set offer security to the players in the sense that each "objection" can be met by a "counter objection." Different concepts of objections and counter objections then, define different notions of stability and hence, different bargaining sets.

An interesting property of the various bargaining sets introduced in [4] is the following. For a given cooperative game in characteristic function form, each bargaining set contains at least one payoff vector which can be considered stable, relative to the concepts of objections and counter objections utilized to define that particular bargaining set. This should not come as a surprise, of course, because according to the above hypothesis one should expect that, at least, the payoff vector where every player receives what he can get on his own is the most secure place to be, by any standards. In fact, for certain games, the only payoff vector belonging to some of the bargaining sets is the one described above.

Although, some may find disturbing the idea that, in some games, the players would settle at an outcome which is not Pareto optimal, if security, is what the players are after, then we may have to accept the fact that Pareto optimality is not a necessary condition for stability. As was pointed out by M. Davis and M. Maschler [8], "often a desire for security is stronger than a wish to make some extra profit."

Now, certain solution concepts, such as the core of a game, exclude the possibility that the players would settle at an outcome which is not Pareto optimal. It would seem then that such solution concepts could be defined in terms of two criteria: (a) a "security criterion," reflecting the safety requirements of the players, and (b) the criterion of Pareto optimality, reflecting the desire of the players to make the most out of a game, which we can refer to as a "maximum benefit criterion."

The two, of course, do not have to be compatible with each other for every game. Therefore, and for certain games, solution concepts defined on the basis of those two criteria could be empty.

Provided that one is willing to accept the hypothesis that certain solution concepts, such as the core of a game, could be defined in terms of the two criteria given above, the following process could be followed concerning the derivation of solution concepts to n-person cooperative games. First, and following Aumann and Maschler's hypothesis, one starts out by defining a "basic solution concept" based solely on the safety requirements of the players. Second, one could proceed one step further and define a new solution concept consisting of the Pareto optimal points of the basic solution concept, derived in the first step.

Applying this process to its full, for example, one could derive new solution concepts out of the Pareto optimal points of the bargaining sets, or the kernel of a game introduced by Davis and Maschler [7]. For one of the bargaining sets in particular, the bargaining set $M_1^{(i)}$, and for the kernel, one will have that the, so derived, new solution concepts will never be empty for classical games. It is well known, Davis and Maschler [8], Peleg [27], Maschler and Peleg [21], Schmeidler [34], that, for classical games, both, the bargaining set $M_1^{(i)}$, and the kernel, always contain Pareto optimal payoff vectors.

What gives credibility to the two-step process described above for deriving certain solution concepts to n -person cooperative games is that the core of a game can be derived by means of that process.

In Section 4.3 below, we introduce a basic solution concept where stability is obtained through the safety requirements of the players (i.e., by following Aumann and Maschler's hypothesis). It is then shown (Theorem 4.3) that the Pareto optimal payoff vectors which are stable in the above sense, constitute the core of the relevant game. For the basic solution concept, stability for a payoff vector means that domination of that payoff vector (by another payoff vector) does not reduce the corresponding individual payoff of any player.

In Section 4.4 the above notion of stability is generalized in the following way. Even if domination by any number of disjoint coalitions could reduce the corresponding payoff of a player at a given payoff vector, that payoff vector could still satisfy the safety requirements of that player if the latter could counteract with a coalition that will guarantee him his original payoff. The payoff vectors that satisfy this notion of stability for a game are called individually defensible positions for that game.

In Section 4.5, and following the two-step process described above, we use the set of individually defensible positions of a game as the basic solution concept to obtain from its Pareto optimal points a generalized concept of the core, the effective core of a game. It is then shown (Theorem 4.5), by using Peleg's method for proving the existence of the bargaining set $M_1^{(1)}$ in [27], that the effective core is not empty for classical games.

In Section 4.6, a comparison of the bargaining theory of Aumann and Maschler [4], with the theory developed here is made. It is shown there that elements of the kernel and the bargaining set $M_1^{(i)}$ of a game are not necessarily individually defensible positions (Theorem 4.6).

Section 4.2, below, deals with notation and preliminary definitions.

4.2 Notation and Preliminary Definitions

An n -person cooperative game in characteristic function form with side payments and transferable utility is defined with the help of the following notation.

Notation

$N \equiv \{1, \dots, n\}$, the set of players.

$C_N \equiv \{c: c \subseteq N\}$, the power set of N ; each element c of C_N is called a coalition.

$C_N^i \equiv \{c \in C_N: i \in c\}$, any $i \in N$, the set of coalitions that contain player i as a member.

$C_N^{i(} \equiv (C_N - C_N^i)$, any $i \in N$.

$\tau \equiv \{c_1, \dots, c_k\}$, a collection of coalitions in C_N .

T_N the set of partitions of the set N , i.e., $T_N \equiv \{\tau: (a) \bigcup_{c_j \in \tau} c_j = N$, and (b) $((c_r, c_s \in \tau, r \neq s) \Rightarrow (c_r \cap c_s = \emptyset))\}$; an element τ of T_N is called a coalition structure.

$v: C_N \rightarrow E^1$ a characteristic function; for each c , $c \in C_N$, we write $v(c)$ to denote the value of coalition c , and we assume that:

(a) $v(\{\emptyset\}) = 0$, and (b) $|v(c)| < \infty$, $\forall c, c \in C_N$.

$\Gamma \equiv (N, v)$ an n -person cooperative game in characteristic function form, with side payments, and transferable utility.

$x = (x^1, \dots, x^n)$ denotes an outcome of Γ ; $x \in E^n$; x^i denotes the i -th element of x .

$(x^i)_j \in c$ denotes the vector of outcomes for the players of a coalition c , given x , $c \in C_N$, $x \in E^n$.

$x(c) \equiv \sum_{i \in c} x^i$, the payoff to coalition c for any outcome x , $x \in E^n$, and any coalition c , $c \in C_N$.

$X_\tau(\Gamma) \equiv \{x \in E^n: (1) x^i \geq v(\{i\}), \forall i, i \in N, \text{ and } (2) x(N) \leq \sum_{c \in \tau} v(c)\}$, the set of feasible, individually rational, outcome vectors of

Γ given the coalition structure τ , $\tau \in T_N$.

$\bar{X}_\tau(\Gamma) \equiv \{x \in X_\tau(\Gamma): x(c) = v(c), \forall c, c \in \tau\}$, for any coalition structure τ , $\tau \in T_N$.

$X(\Gamma) \equiv \bigcup_{\tau \in T_N} X_\tau(\Gamma)$, the set of payoff vectors of Γ .

$X^*(\Gamma) \equiv \{x^* \in X(\Gamma): x^*(N) = \text{Maximum}_{\tau \in T_N} \{ \sum_{c \in \tau} v(c) \}$, the set of Pareto optimal payoff vectors of Γ .

$PC(\Gamma) \equiv \bigcap_{\tau \in T_N} (\bar{X}_\tau(\Gamma) \times \tau)$, where

$\bigcap_{\tau \in T_N} (\bar{X}_\tau(\Gamma) \times \tau) = \{(x; \tau): x \in \bar{X}_\tau(\Gamma), \tau \in T_N\}$, the set of individually rational payoff configurations (i.r.p.c.) of Γ .

Definition 4.1: Given an n -person cooperative game $\Gamma = (N, v)$, let

$\tilde{x}$, $x \in X(\Gamma)$, and let $\tilde{c}$, $\tilde{c} \in C_N$, be such that $\tilde{c} \neq \emptyset$. We say that the payoff vector $\tilde{x}$ dominates the payoff vector x via coalition $\tilde{c}$, and

we write: $\tilde{x} \text{ dom}_{\tilde{c}} x$, if, and only if. (a) $v(\tilde{c}) \geq \tilde{x}(\tilde{c})$, and (b)

$$(\tilde{x}^i)_{i \in \tilde{c}} \geq (x^i)_{i \in \tilde{c}}.$$

For any two payoff vectors $\tilde{x}$, $x \in X(\Gamma)$, we define:

$$(4.1) \quad C_\Gamma(\tilde{x} \text{ d } x) \equiv \{\tilde{c} \in C_N: \tilde{x} \text{ dom}_{\tilde{c}} x\}.$$

The (usual) binary relationship "domination" over the set $X(\Gamma)$ is defined as follows.

Definition 4.2: Given an n -person cooperative game $\Gamma = (N, v)$, let $\tilde{x}, x \in X(\Gamma)$. We say that the payoff vector $\tilde{x}$ dominates the payoff vector x (without reference to any specific coalition) and we write: $\tilde{x} \text{ dom } x$, if and only if $C_\Gamma(\tilde{x} \text{ d } x) \neq \emptyset$. (I.e., $(\tilde{x}, x \in X(\Gamma), \tilde{x} \text{ dom } x) \iff (C_\Gamma(\tilde{x} \text{ d } x) \neq \emptyset)$.)

All payoff vectors $x, x \in X(\Gamma)$, which are not dominated, in the sense of the binary relationship dom, defined above, by any payoff vector $\tilde{x}, \tilde{x} \in X(\Gamma)$, constitute the (usual) core of Γ . In particular, for any payoff vector $x, x \in X(\Gamma)$, let

$$(4.2) \quad B_\Gamma(x) \equiv \{\tilde{x} \in X(\Gamma): \tilde{x} \text{ dom } x\}.$$

Definition 4.3: Let $\Gamma = (N, v)$. The set $K(\Gamma)$, $K(\Gamma) \equiv \{x \in X(\Gamma): B_\Gamma(x) = \emptyset\}$, is the core of the game Γ .

Alternatively, if for any payoff vector $x, x \in X(\Gamma)$, we define:

$$(4.3) \quad C_\Gamma(dx) \equiv \bigcup_{\tilde{x} \in X(\Gamma)} C_\Gamma(\tilde{x} \text{ d } x)$$

then the core of Γ can be defined as

$$(4.4) \quad K(\Gamma) \equiv \{x \in X(\Gamma): C_\Gamma(dx) = \emptyset\}.$$

The definition of the core of a game Γ given in (4.4) is important for the following reason. Suppose that $C_\Gamma(dx) = \emptyset$, for some payoff vector $x, x \in X(\Gamma)$. Then, and according to (4.3), $C_\Gamma(\tilde{x} \text{ d } x) = \emptyset, \forall \tilde{x} \in X(\Gamma)$, and this must hold true, in particular, $\forall \tilde{x}, \tilde{x} \in X(\Gamma)$, such that $\tilde{x}(c) = v(c)$,

for some coalition c , $c \in C_N$. But for each non empty coalition c , $c \in C_N$, there exists a payoff vector $\tilde{x}$, $\tilde{x} \in X(\Gamma)$, such that $\tilde{x}(c) = v(c)$. It will follow then that

$$(4.5) \quad (C_\Gamma(dx) = \emptyset) \iff (x(c) \geq v(c), \forall c, c \in C_N, \exists: c \neq \emptyset).$$

But (4.5), in conjunction with (4.4), implies that

$$(4.6) \quad K(\Gamma) = \{x \in X(\Gamma): x(c) \geq v(c), \forall c, c \in C_N, \exists: c \neq \emptyset\}.$$

A payoff vector x , $x \in X(\Gamma)$, that satisfies the properties on the right hand side of (4.6) is called group rational. Expression (4.6) then, is the familiar definition of the core of a game Γ , as the set of payoff vectors in $X(\Gamma)$ that satisfy the criterion of group rationality.

4.3 Individually Strongly Defensible Positions

For a given game $\Gamma = (N, v)$ a situation may arise where for a given payoff vector there is no concerted action of any groups of players that could affect adversely the corresponding payoff of any particular player not belonging to those groups. Such payoff vectors enjoy a special kind of stability which we explore in this section.

Given a game Γ , for any payoff vector x , $x \in X(\Gamma)$, and for any player i , $i \in N$, we denote by $B_\Gamma^{i-}(x)$ the set of payoff vectors in $X(\Gamma)$ that: (a) dominate x , and (b) such domination reduces the payoff corresponding to player i , in x , i.e.,

$$(4.7) \quad B_\Gamma^{i-}(x) \equiv \{\tilde{x} \in B_\Gamma(x): \tilde{x}^i < x^i\}.$$

Suppose now that player i considers the payoff vector x , and he is concerned whether he can enjoy his own payoff x^i without any disturbances or threats from any group or groups of players that do not include him.

Then, he must be concerned with the set $B_{\Gamma}^{i-}(x)$. In particular, if the set $B_{\Gamma}^{i-}(x)$ is empty, player i could find the payoff vector x as a secure place to be since domination does not reduce his own payoff x^i . In fact, if x is such that the sets $B_{\Gamma}^{i-}(x)$ are empty for every player i , $i \in N$, every player could feel secure at x , in the above sense. Therefore, for a given game Γ , and any payoff vector x , $x \in X(\Gamma)$, the criterion: "domination does not reduce the payoff to any player at x " is a stability criterion for payoff vectors in $X(\Gamma)$. The corresponding solution concept is given in the following definition.

Definition 4.4: Let $\Gamma = (N, v)$.

- (a) A payoff vector x , $x \in X(\Gamma)$ is a strongly defensible position, in Γ , for player i , $i \in N$, if and only if $B_{\Gamma}^{i-}(x) = \emptyset$.
- (b) $SD^i(\Gamma) \equiv \{x \in X(\Gamma) : B_{\Gamma}^{i-}(x) = \emptyset\}$, any $i \in N$.
- (c) $SD(\Gamma) \equiv \bigcap_{i \in N} SD^i(\Gamma)$.
- (d) A payoff vector x , $x \in X(\Gamma)$, is an individually strongly defensible position in Γ if and only if $x \in SD(\Gamma)$.

As to what payoff vectors in $X(\Gamma)$ constitute strongly defensible positions for a player i , $i \in N$, Theorem 4.1 below reveals that the set $SD^i(\Gamma)$ consists of those payoff vectors in $X(\Gamma)$ where (a) player i receives exactly what he can get on his own, or (b) every coalition that does not include player i receives at least its value.

Theorem 4.1: For a given game $\Gamma = (N, v)$, and for each player i , $i \in N$,

$$SD^i(\Gamma) = \{x \in X(\Gamma) : (1) x^i = v(\{i\}) \text{ or } (2) x(c) \geq v(c), \forall c \in C_N^{i(-)}\}.$$

Proof: Observe that for any player i , $i \in N$,

$$(4.8) \quad (x \in X(\Gamma), \text{ and } x^i = v(\{i\})) \Rightarrow (B_{\Gamma}^{i-}(x) = \emptyset).$$

On the other hand,

$$(4.9) \quad (x \in X(\Gamma), \text{ and } x(c) \geq v(c) \quad \forall c, c \in C_N^{i(}, \\ \text{some } i \in N) \Rightarrow (B_{\Gamma}^{i-}(x) = \emptyset),$$

i.e., if for any player i , $i \in N$, and if for some payoff vector x , $x \in X(\Gamma)$, we have that $x(c) \geq v(c)$, $\forall c \in C_N^{i(}$, then either $B_{\Gamma}(x) = \emptyset$ (hence $B_{\Gamma}^{i-}(x) = \emptyset$) or $B_{\Gamma}(x) \neq \emptyset$ and $C_{\Gamma}(dx) \subset C_N^i$ (hence, again, $B_{\Gamma}^{i-}(x) = \emptyset$).

As a consequence of (4.8) and (4.9), the proof of the theorem will be complete if we can show that for each player i , $i \in N$,

$$(4.10) \quad (x \in SD^i(\Gamma), \text{ and } x^i > v(\{i\})) \Rightarrow (x(c) \geq v(c) \quad \forall c \in C_N^{i(}).$$

For the sake of the argument suppose that there exists a player i , $i \in N$, and there exists a payoff vector x , $x \in SD^i(\Gamma)$ such that: (a) $x^i > v(\{i\})$ and (b) $x(\tilde{c}) < v(\tilde{c})$ for some coalition $\tilde{c} \in C_N^{i(}$. From the set of payoff vectors $\tilde{x}$, $\tilde{x} \in X(\Gamma)$, that satisfy the condition: $\tilde{x}(\tilde{c}) = v(\tilde{c})$, we can select one, say $\tilde{x}^*$ that will also satisfy: (a) $(\tilde{x}^{*j})_{j \in \tilde{c}} \geq (x^j)_{j \in \tilde{c}}$, and (b) $\tilde{x}^{*s} = v(\{s\}) \quad \forall s, s \in N, \exists: s \notin \tilde{c}$. Apparently, $\tilde{x}^* \underline{\text{dom}}_c x$, and $\tilde{x}^* \in B_{\Gamma}^{i-}(x)$, which contradicts the fact that $x \in SD^i(\Gamma)$ implies that $B_{\Gamma}^{i-}(x) = \emptyset$.

q.e.d.

As a corollary to Theorem 4.1 it is easy to show that payoff vectors which are strongly defensible positions for a player i and give him more than what he can get on his own, are dominated, if at all, only via coalitions that contain player i . Formally, we have.

Corollary 4.1: For a given game $\Gamma = (N, v)$, and any player i , $i \in N$,

$$((x \in SD^1(\Gamma), x^i > v(\{i\}), \text{ and } B_\Gamma(x) \neq \emptyset) \Rightarrow (C_\Gamma(dx) \subseteq C_N^i)).$$

Another conclusion that follows from Theorem 4.1 is that the set of individually strongly defensible positions of a game Γ , $SD(\Gamma)$, is never empty. Apparently, the payoff vector x , where $x^i = v(\{i\})$, $\forall i \in N$, belongs to the set $SD(\Gamma)$, for any game $\Gamma = (N, v)$.

The next theorem shows that if the core of a game Γ , is empty then, the only payoff vector that constitutes an individually strongly defensible position for that game is the one where every player receives what he can get on his own.

Theorem 4.2: Let $\Gamma = (N, v)$ be such that $K(\Gamma) = \emptyset$. Then,

$$SD(\Gamma) = \{(v(\{i\}))_{i \in N}\}.$$

Proof: We need only show that $K(\Gamma) = \emptyset$ implies that $SD(\Gamma) \subseteq \{(v(\{i\}))_{i \in N}\}$.

(I.e., the converse will follow trivially from the fact that any x , $x \in X(\Gamma)$, such that $x^i = v(\{i\})$, for any i , $i \in N$, is such that $x \in SD^1(\Gamma)$.) Suppose to the contrary that $K(\Gamma) = \emptyset$ and that there exists a payoff vector x , $x \in SD(\Gamma)$, such that $x \neq (v(\{i\}))_{i \in N}$.

Given any x , as above, consider the coalition structure $\bar{\tau}$, $\bar{\tau} \in T_N$, $\bar{\tau} = \{\bar{c}, \hat{c}\}$, where

$$\bar{c} = \{j \in N: x^j > v(\{j\})\}, \text{ and}$$

$$\hat{c} = \{s \in N: x^s = v(\{s\})\}.$$

The hypothesis that $x \neq (v(\{i\}))_{i \in N}$, if correct, implies that $\bar{c} \neq \emptyset$.

On the other hand, the assumption that $K(\Gamma) = \emptyset$ implies that $B_\Gamma(x) \neq \emptyset$.

Hence, by Corollary 4.1, for each $j \in \bar{c}$:

$$(4.11) \quad C_{\Gamma}(dx) \subset C_N^j.$$

This means that the only coalitions that dominate the payoff vector x are those that include every player $j \in \bar{c}$. Stated otherwise, (4.11) means that

$$(4.12) \quad x(c) \geq v(c), \forall c, c \in C_N, \quad \exists: c \not\supseteq \bigcap_{j \in \bar{c}} C_N^j.$$

For that matter, for any payoff vector $\tilde{x}$, $\tilde{x} \in X(\Gamma)$, such that

$$(\tilde{x}^i)_{i \in N} \geq (x^i)_{i \in N}, \text{ we must also have that}$$

$$(4.13) \quad \tilde{x}(c) \geq v(c), \forall c, c \in C_N, \quad \exists: c \not\supseteq \bigcap_{j \in \bar{c}} C_N^j.$$

Consider now any Pareto optimal payoff vector $\tilde{x}^*$ that satisfies conditions (4.14) - (4.16) below.

$$(4.14) \quad \tilde{x}^{*s} = v(\{s\}), \forall s \in \hat{c};$$

$$(4.15) \quad (\tilde{x}^{*j})_{j \in \bar{c}} \geq (x^j)_{j \in \bar{c}}; \text{ and}$$

$$(4.16) \quad \tilde{x}^*(c) \geq v(c), \forall c \in \bigcap_{j \in \bar{c}} C_N^j.$$

Note that the existence of a Pareto optimal payoff vector $\tilde{x}^*$ that satisfies (4.14) and (4.15) is guaranteed by (4.11). That any such payoff vector will also satisfy (4.16) will follow from the Pareto optimality of $\tilde{x}^*$. More specifically, given any Pareto optimal payoff vector $\tilde{x}^*$ that satisfies (4.14), we must have that $\tilde{x}^*(\bar{c}) = \tilde{x}^*(N) - \tilde{x}^*(\hat{c})$. Because $\tilde{x}^*(N)$, and $\tilde{x}^*(\hat{c})$ are constant in a given game $\Gamma = (N, v)$, so is $\tilde{x}^*(\bar{c})$. Therefore, if there exists a coalition c^* , $c^* \in \bigcap_{j \in \bar{c}} C_N^j$, that violates (4.16), e.g., if $\tilde{x}^*(c^*) < v(c^*)$, then we will have a contradiction.

Let us observe now that the payoff vector $\tilde{x}^*$ that satisfies (4.14) - (4.16) also satisfies (4.13). But then, $\tilde{x}^* \in K(\Gamma)$, a contradiction because we have assumed that $K(\Gamma) = \emptyset$.

q.e.d.

Let us proceed now to show that the hypothesis, stated in the introduction of this chapter, that certain solution concepts to n-person cooperative games can be defined in terms of a "security criterion" and a "maximum benefit criterion" holds at least for the core. Using the requirement that a payoff vector must be an individually strongly defensible position, as a security criterion, and Pareto optimality as a maximum benefit criterion we prove

Theorem 4.3: $K(\Gamma) = (X^*(\Gamma) \cap SD(\Gamma))$, for any game $\Gamma = (N, v)$.

Proof:

(a) $K(\Gamma) \subseteq (X^*(\Gamma) \cap SD(\Gamma))$: Either $K(\Gamma) = \emptyset$, or $K(\Gamma) \neq \emptyset$. If $K(\Gamma) = \emptyset$, we must have that,

$$(4.17) \quad (K(\Gamma) = \emptyset) \Rightarrow (B_T(x) \neq \emptyset, \forall x \in X(\Gamma)).$$

Hence, if $\bar{x} \equiv (v(\{i\}))_{i \in N}$, then $\bar{x} \notin X^*(\Gamma)$. By Theorem 4.1,

$$(4.18) \quad (K(\Gamma) = \emptyset) \Rightarrow (SD(\Gamma) = \{\bar{x}\}).$$

It will follow then that,

$$(4.19) \quad (K(\Gamma) = \emptyset) \Rightarrow (X^*(\Gamma) \cap SD(\Gamma) = \emptyset).$$

Suppose now that $K(\Gamma) \neq \emptyset$, and that x is an arbitrary element of $K(\Gamma)$.

Then,

$$(4.20) \quad (x \in K(\Gamma)) \iff ((a) \ x \in X^*(\Gamma), \text{ and } (b) \ B_\Gamma(x) = \emptyset).$$

But,

$$(4.21) \quad (B_\Gamma(x) = \emptyset) \Rightarrow (B_\Gamma^{i-}(x) = \emptyset, \forall i \in N),$$

while, from Definition 4.4, for any player i , $i \in N$,

$$(4.22) \quad (B_\Gamma^{i-}(x) = \emptyset) \Rightarrow (x \in SD^i(\Gamma)).$$

It will follow then that

$$(4.23) \quad (K(\Gamma) \neq \emptyset) \Rightarrow (K(\Gamma) \subseteq (X^*(\Gamma) \cap SD(\Gamma))).$$

Therefore, (4.19) and (4.23) imply that

$$(4.24) \quad K(\Gamma) \subseteq (X^*(\Gamma) \cap SD(\Gamma)).$$

(b) $\underline{(X^*(\Gamma) \cap SD(\Gamma)) \subseteq K(\Gamma)}$: Again, either $(X^*(\Gamma) \cap SD(\Gamma)) = \emptyset$, or $(X^*(\Gamma) \cap SD(\Gamma)) \neq \emptyset$. Observe that,

$$(4.25) \quad ((X^*(\Gamma) \cap SD(\Gamma)) = \emptyset) \Rightarrow (\text{For each } x, x \in X^*(\Gamma), \exists j, j \in N,$$

$$\exists: B_\Gamma^{j-}(x) \neq \emptyset).$$

Hence,

$$(4.26) \quad ((X^*(\Gamma) \cap SD(\Gamma)) = \emptyset) \Rightarrow (B_\Gamma(x) \neq \emptyset, \forall x \in X^*(\Gamma)).$$

Therefore,

$$(4.27) \quad ((X^*(\Gamma) \cap SD(\Gamma)) = \emptyset) \Rightarrow (K(\Gamma) = \emptyset).$$

Suppose now that $(X^*(\Gamma) \cap SD(\Gamma)) \neq \emptyset$, and that x is an arbitrary element of the set $(X^*(\Gamma) \cap SD(\Gamma))$. Observe that for x , as above, $x \in X^*(\Gamma)$ and $x \in SD(\Gamma)$. By Definition 4.4,

$$(4.28) \quad (x \in SD(\Gamma)) \Rightarrow (B_T^{i-}(x) = \emptyset, \forall i \in N).$$

But (4.28), in conjunction with the fact that x is Pareto optimal, is sufficient to show that

$$(4.29) \quad x \in K(\Gamma).$$

Because x was chosen arbitrarily from the set $(X^*(\Gamma) \cap SD(\Gamma))$, we must have

$$(4.30) \quad (X^*(\Gamma) \cap SD(\Gamma)) \subseteq K(\Gamma).$$

q.e.d.

As for the topological properties of the sets $SD^i(\Gamma)$, $i \in N$, and the set $SD(\Gamma)$ we prove the following lemma.

Lemma 4.1: Let $\Gamma = (N, v)$. Then, the sets $SD(\Gamma)$, and $SD^i(\Gamma)$, $\forall i \in N$, are closed in $X(\Gamma)$ and compact.

Proof: Apparently, $SD(\Gamma)$ will be closed and compact if for each i , $i \in N$, $SD^i(\Gamma)$ is closed and compact, since $SD(\Gamma)$ consists of the intersection of the sets $SD^i(\Gamma)$, $\forall i \in N$.

Given a payoff vector x , $x \in X(\Gamma)$, and any positive real number r , let

$$(4.31) \quad Nd_r(x) \equiv \{\tilde{x} \in X(\Gamma): d(x, \tilde{x}) < r\},$$

where, as may be recalled, $d(x, \tilde{x})$ denotes the Euclidean distance between the payoff vectors x and $\tilde{x}$. The set $Nd_r(x)$ then, is an open neighborhood for x , for a given r . Therefore, a subset A , $A \subseteq X(\Gamma)$, will be open in $X(\Gamma)$ if for each x , $x \in A$, there exists a positive real number r such that $Nd_r(x) \subseteq A$.

We shall show that each set $SD^i(\Gamma)$ is closed in $X(\Gamma)$ by showing that $SD^i(\Gamma)$ consists of the union of two closed sets.

From Theorem 4.1, we can write, for each $i \in N$, the set $SD^i(\Gamma)$ as $SD^i(\Gamma) = (A^i \cup B^i)$ where

$$A^i \equiv \{x \in X(\Gamma): x^i = v(\{i\})\}, \text{ and}$$

$$B^i \equiv \{x \in X(\Gamma): x(c) \geq v(c), \forall c \in C_N^i\}.$$

To show that A^i is closed in $X(\Gamma)$, observe that its complement

$(X(\Gamma) - A^i)$ is open in $X(\Gamma)$, because if $x \in (X(\Gamma) - A^i)$ and if:

$$r = \min_{\tilde{x} \in A^i} \{d(x, \tilde{x})\} \text{ then } Nd_r(x) \subseteq (X(\Gamma) - A^i).$$

To show that B^i is closed in $X(\Gamma)$ let $x \in (X(\Gamma) - B^i)$. Then, there exists a coalition c , $c \in C_N^i$, such that $x(c) < v(c)$, which follows from the definition of the set B^i . Furthermore, if $\tilde{x}$, $\tilde{x} \in X(\Gamma)$, is such that $\tilde{x}(c) = v(c)$, then $d(x, \tilde{x}) > 0$. Let r be such that

$$(4.32) \quad r = \min \{d(x, \tilde{x}): (a) \tilde{x} \in X(\Gamma), \text{ and } (b) \tilde{x}(c) = v(c)$$

$$\text{for any } c, c \in C_N^i, \exists: x(c) < v(c)\}.$$

Note that if $\tilde{x}^*$ and $\tilde{c}^*$ solve the minimization problem implicit in (4.32), then, for any coalition $\bar{c}$, $\bar{c} \in C_N^i$, such that $\bar{c} \neq \tilde{c}^*$, and such that

$x(\bar{c}) < v(\bar{c})$, and for any payoff vector $\bar{x}$, $\bar{x} \in X(\Gamma)$, such that $\bar{x}(\bar{c}) = v(\bar{c})$, we must have that $d(x, \bar{x}^*) \leq d(x, \bar{x})$. Therefore, $Nd_r(x) \subseteq (X(\Gamma) - B^i)$. Because x was arbitrarily chosen, the set $(X(\Gamma) - B^i)$ is open and the set B^i is closed in $X(\Gamma)$. This completes the proof that for each player i , $i \in N$, the set $SD^i(\Gamma) = (A^i \cup B^i)$ is closed in $X(\Gamma)$. The compactness of each set $SD^i(\Gamma)$, $i \in N$, and the set $SD(\Gamma)$ then follows from the compactness of $X(\Gamma)$.

q.e.d.

4.4 Individually Defensible Positions

The notion of stability introduced in Section 4.3 relies upon the assumption that each player will be willing to settle at a payoff vector if he is not threatened as far as his own payoff is concerned. As such, it is a very strong notion of stability because it ignores the fact that a player, possibly, could counteract the actions of any group or groups of players that do not contain him. Such actions and counter actions are considered in this section.

Given a game $\Gamma = (N, v)$, we denote by $F(\Gamma)$ the set of payoff vectors that do not constitute strongly defensible positions for any player i , $i \in N$, i.e.,

$$(4.33) \quad F(\Gamma) \equiv (X(\Gamma) - \bigcup_{i \in N} SD^i(\Gamma)).$$

The closure of the set $F(\Gamma)$ will be denoted by $\bar{F}(\Gamma)$, i.e.,

$$(4.34) \quad \bar{F}(\Gamma) \equiv \{x \in X(\Gamma) : Nd_r(x) \cap F(\Gamma) \neq \emptyset, \\ \forall r, r \in E^1, r > 0\}.$$

The set $\overline{F}(\Gamma)$ then is the smallest closed set in $X(\Gamma)$ that contains the set $F(\Gamma)$.

The following remark about the sets $F(\Gamma)$ and $\overline{F}(\Gamma)$ is stated for future reference.

REMARK: By Lemma 4.1, for each player i , $i \in N$, the set $SD^i(\Gamma)$ is closed in $X(\Gamma)$. It will follow then that the set $F(\Gamma)$ is open in $X(\Gamma)$. On the other hand if $Br(F(\Gamma))$ denotes the boundary of $F(\Gamma)$, the fact that $F(\Gamma)$ is open will imply that: $Br(F(\Gamma)) = (\overline{F}(\Gamma) - F(\Gamma))$. From (4.33), payoff vectors in $Br(F(\Gamma))$ must constitute strongly defensible positions in Γ for at least one player i , $i \in N$. However, the elements of the set $Br(F(\Gamma))$ cannot be interior points of any set $SD^i(\Gamma)$, $i \in N$.

Next, for any player i , $i \in N$, let us denote by T_N^i the set of coalition structures in N that contain player i as a single person coalition, i.e.,

$$(4.35) \quad T_N^i \equiv \{\tau \in T_N: \{i\} \in \tau\}.$$

In order to indicate that a coalition structure τ belongs to the set T_N^i we will add the subscript i after τ , i.e., the elements of T_N^i are denoted by τ_i .

Suppose now that some player i , $i \in N$, considers a payoff vector x , $x \in \overline{F}(\Gamma)$, and he is concerned about his own payoff x^i . Apparently, if $x \in SD^i(\Gamma)$ he knows that he can enjoy x^i without any disturbances or threats from any group or groups of players that do not contain him. However, if $x \notin SD^i(\Gamma)$ he may be faced with the following situation. The players in the set $N - \{i\}$ form disjoint coalitions, some of which

have to gain by moving away from x , and decide on a distribution of the value of such coalitions, leaving player i on his own. Faced with such concerted action by the others, player i may counter with an action of his own. For example, he could try to form a coalition with some of the players and move away from the position where his payoff is what he can get on his own. If by such a counter action to each "reasonable" action of $N - \{i\}$, player i can regain his original payoff x^i , we say that x is a "weakly defensible position for player i ." Formally, we have

Definition 4.5: Let $\Gamma = (N, v)$.

(a) A payoff vector x , $x \in X(\Gamma)$, is a weakly defensible position, in Γ , for player i , $i \in N$, if, and only if,

(a.1) $x \in \overline{B}(\Gamma)$, and $B_{\Gamma}^{i-}(x) \neq \emptyset$; and

(a.2) for each individually rational payoff configuration

$(\tilde{x}; \tilde{\tau}_1)$, such that

(i) $\tilde{x} \in B_{\Gamma}^{i-}(x)$, and $\tilde{\tau}_1 \in T_N^i$,

(ii) $(\tilde{x}^j)_{j \in \tilde{c}} \geq (x^j)_{j \in \tilde{c}}$, $\forall \tilde{c}$, $\tilde{c} \in (\tilde{\tau}_1 \cap C_{\Gamma}(dx))$, and

(iii) $(\tilde{x}^j)_{j \in \tilde{c}} \leq (x^j)_{j \in \tilde{c}}$, $\forall \tilde{c}$, $\tilde{c} \in \tilde{\tau}_1$, $\exists: \tilde{c} \notin C_{\Gamma}(dx)$,

there exists a coalition $\hat{c}$, $\hat{c} \in C_N^i$, and there exists a payoff vector $\hat{x}$, $\hat{x} \in X(\Gamma)$, such that

(iv) $\hat{x}(\hat{c}) = v(\hat{c})$,

(v) $\hat{x}^j \geq \tilde{x}^j$, $\forall j$, $j \in \hat{c}$, and

(vi) $\hat{x}^i \geq x^i$.

(b) $WD^i(\Gamma) \equiv \{x \in X(\Gamma): x \text{ is a weakly defensible position for player } i\}$, $i \in N$.

For the class of n -person cooperative games considered in this chapter, the existence part of condition (a.2), in the above definition, could be restated as follows: "there exists a coalition $\hat{c}$, $\hat{c} \in C_N^i$, such that $v(\hat{c}) \geq \bar{x}(\hat{c} - \{i\}) + x^i$, where, $(\hat{c} - \{i\}) = \{j \in \hat{c} : j \neq i\}$."

In words, condition (a.1) says that x is not a strongly defensible position for player i , and if there exists a player j , $j \neq i$, such that $x \in SD^j(\Gamma)$ then (see remark on page 115) $x \in Br(SD^j(\Gamma))$. Using the closure of $F(\Gamma)$, $\bar{F}(\Gamma)$, instead of the set $F(\Gamma)$ itself, allows the possibility that a payoff vector x , $x \in Br(\bar{F}(\Gamma))$, could be a weakly defensible position for one player and a strongly defensible position for another. This is done in order to make the relevant sets closed in $X(\Gamma)$.

The choice of a payoff configuration $(\tilde{x}; \tilde{\tau}_i)$ for the "threat" against player i is designed to reflect the "worst possible situation that player i may have to face." Given a payoff vector x , $x \in X(\Gamma)$, one could explain the formation of the coalition structure $\tilde{\tau}_i$ in terms of one of several processes. For example, one could start out by saying that the coalition with the greatest excess over x (or the greatest per player excess over x) is formed. Given that such a coalition has been formed, the disjoint coalition with the next higher excess over x (or the next higher per player excess over x) forms. The process is repeated until player i is left alone. This last condition, i.e., that player i will be left alone may sound unreasonable. However, one should remember that what we are after, is the possibility that player i may have to face the worst possible situation, given the rules of the game. Presumably, if a player can counter an action of the other players that affects him more adversely, he could also counter a less severe action.

Conditions (ii) and (iii) imposed on the payoff vector $\tilde{x}$ are "distribution requirements" for the value of each coalition in the coalition structure $\tilde{\tau}_1$. In words, these distribution requirements say the following. No coalition in $\tilde{\tau}_1$ that stands to gain in moving away from x should make any of its players worse off. On the other hand, no coalition in $\tilde{\tau}_1$ that does not stand to gain in moving away from x should make any of its players better off.

A payoff configuration $(\tilde{x}; \tilde{\tau}_1)$ that satisfies requirements (i) - (iii) in (a.2), then, can be called a reasonable action of the players in $N - \{i\}$ against player i .

The conditions imposed on the payoff vector $\hat{x}$ and coalition $\hat{c}$ could be summarized as saying that the value of coalition $\hat{c}$, $v(\hat{c})$, is sufficient to compensate every player $j \in \hat{c}$ in moving away from $\tilde{x}$, while player i can regain his payoff corresponding to x , x^i . It is worth noting that since $(\tilde{x}; \tilde{\tau}_1)$ is an i.r.p.c. such that $\{i\} \in \tilde{\tau}_1$, then $\tilde{x}^i = v(\{i\})$. This, in conjunction with the fact that $\tilde{x} \in B_N^{1-}(x)$, is sufficient to show that if a payoff vector $\hat{x}$ and a coalition $\hat{c}$, that satisfy conditions (iv)-(vi) of part (a.2), in Definition 4.5, exist, then $\hat{x} \underline{\text{dom}}_c \tilde{x}$.

In view of the interpretation of a weakly defensible position for a player i , $i \in N$, (viz., that for each reasonable action by the players in $N - \{i\}$ there is a counteraction by player i that will preserve player i 's payoff in that position) individually strongly defensible positions take a special meaning. A strongly defensible position for player i is one where the set of reasonable actions by the players in $N - \{i\}$ is empty. Therefore, these two concepts can be incorporated into one, more general, criterion of stability. This criterion will say that a payoff vector x is

"stable" if it is either a strongly or a weakly defensible position for each player. Payoff vectors which can be considered stable in this sense, are called: "individually defensible positions." Formally, we have

Definition 4.6: Let $\Gamma = (N, v)$.

- (a) A payoff vector x , $x \in X(\Gamma)$, is a defensible position, in Γ , for player i , $i \in N$, if and only if: $x \in (SD^i(\Gamma) \cup WD^i(\Gamma))$.
- (b) $D^i(\Gamma) \equiv (SD^i(\Gamma) \cup WD^i(\Gamma))$, any $i \in N$.
- (c) $D(\Gamma) \equiv \bigcap_{i \in N} D^i(\Gamma)$.
- (d) A payoff vector x , $x \in X(\Gamma)$, is an individually defensible position in Γ if and only if: $x \in D(\Gamma)$.

Apparently, the set of individually defensible positions of a game Γ is not empty because, as we have pointed out in Section 4.3, the set $SD(\Gamma)$ is not empty. Whether the set $D(\Gamma)$ contains any Pareto optimal payoff vectors is a question that we shall answer in the next section.

The process utilized in this section, and in Section 4.3, for defining individually defensible positions for a game Γ induces a binary relationship over the set $X(\Gamma)$. This binary relationship is obtained by imposing the restrictions of part (a) of Definition 4.5 on the usual domination relationship defined in Section 4.2. Formally, we have

Definition 4.7: Let $\Gamma = (N, v)$, and let $\tilde{x}$, $x \in X(\Gamma)$.

- (a) We say that the payoff vector $\tilde{x}$ effectively dominates the payoff vector x , and we write: $\tilde{x} \text{ e-dom } x$, if and only if there exists a player i , $i \in N$, such that

- (a.1) for some coalition structure $\tilde{\tau}_i, \tilde{\tau}_i \in T_N^i, (\tilde{x}; \tilde{\tau}_i)$ is an i.r.p.c.
- (a.2) $\tilde{x} \in B_\Gamma^{i-}(x)$, and
- (a.3) if $x \in \overline{F}(\Gamma)$, and if x and $(\tilde{x}; \tilde{\tau}_i)$ satisfy conditions (i) - (iii), in part (a.2), of Definition 4.5 then:
 $\tilde{x}(\hat{c} - \{i\}) + x^i > v(\hat{c}), \forall \hat{c} \in C_N^i.$
- (b) $EB_\Gamma^i(x) \equiv \{(\tilde{x}; \tilde{\tau}_i) \in PQ(\Gamma): \tilde{x} \in B_\Gamma^{i-}(x), \tilde{\tau}_i \in T_N^i, \text{ and condition (a.3), above, is satisfied}\}, \text{ any } i \in N.$
- (c) $EB_\Gamma(x) \equiv \bigcup_{i \in N} EB_\Gamma^i(x).$

Because the notion of domination introduced in the above definition is designed to reflect what it means for a payoff vector not to be an individually defensible position for a game Γ , such notion of domination makes sense only in reference to some individual player i . It will follow then that the sets $EB_\Gamma^i(x)$ are well defined sets for each player $i \in N$, and for each payoff vector $x \in X(\Gamma)$. Let us observe that if $B_\Gamma^{i-}(x) \neq \emptyset$, for some $i \in N$, and some $x \in X(\Gamma)$, then there exists a payoff vector $\tilde{x} \in B_\Gamma^{i-}(x)$ that will satisfy conditions (a.1) and (a.2) of the above definition. Hence, the requirement in (a.1) is just a way for establishing some uniformity in conditions (a.2) and (a.3).

To establish the equivalence between the set of individually defensible positions of a game Γ , $D(\Gamma)$, and the set of payoff vectors in $X(\Gamma)$ that are not effectively dominated by any other payoff vectors in $X(\Gamma)$ we prove the following theorem.

Theorem 4.4: Let $\Gamma = (N, v)$. Then,

$$D(\Gamma) = \{x \in X(\Gamma): EB_\Gamma(x) = \emptyset\}$$

Proof: Recall that, by Definition 4.4, for any player i , $i \in N$, $SD^i(\Gamma) \equiv \{x \in X(\Gamma) : B_{\Gamma}^{i-}(x) = \emptyset\}$. It will follow then, from condition (a.2) of Definition 4.7, that for any player i , $i \in N$,

$$(4.36) \quad (x \in SD^i(\Gamma)) \Rightarrow (EB_{\Gamma}^i(x) = \emptyset).$$

On the other hand, for any player i , $i \in N$, for whom $x \in WD^i(\Gamma)$, condition (a.3) of Definition 4.7, cannot be satisfied. Therefore,

$$(4.37) \quad (x \in WD^i(\Gamma)) \Rightarrow (EB_{\Gamma}^i(x) = \emptyset).$$

But for each player i , $i \in N$, $D^i(\Gamma) = (SD^i(\Gamma) \cup WD^i(\Gamma))$. Hence, from (4.36) and (4.37) we must have that

$$(4.38) \quad (x \in D^i(\Gamma)) \Rightarrow (EB_{\Gamma}^i(x) = \emptyset).$$

and, because $D(\Gamma) = \bigcap_{i \in N} D^i(\Gamma)$,

$$(4.39) \quad (x \in D(\Gamma)) \Rightarrow (EB_{\Gamma}(x) = \emptyset).$$

Now, it is easily seen that the converse of (4.39), i.e., that

$$(4.40) \quad (EB_{\Gamma}(x) = \emptyset) \Rightarrow (x \in D(\Gamma)),$$

also holds true. From Definition 4.7,

$$(4.41) \quad (EB_{\Gamma}(x) = \emptyset) \Leftrightarrow (EB_{\Gamma}^i(x) = \emptyset, \forall i \in N).$$

Therefore, if for some x , $x \in X(\Gamma)$, we have that $EB_{\Gamma}(x) = \emptyset$, then, for each payoff vector $\tilde{x}$, $\tilde{x} \in X(\Gamma)$, at least one of the conditions (a.1) - (a.3) of Definition 4.7 must be violated, for each player $i \in N$. Because condition (a.1) cannot be violated without the violation of (a.2) or

(a.3), we must have the following. If (a.2) is violated for any player $i \in N$, $B_i^{1-}(x) = \emptyset$, and $x \in SD^i(\Gamma)$. If (a.3) is violated, for any player $i \in N$, $x \in WD^i(\Gamma)$.

q.e.d.

As for the topological properties of the sets $D^i(\Gamma)$, $i \in N$, and $D(\Gamma)$ we prove the following lemma.

Lemma 4.2: Let $\Gamma = (N, v)$. Then, the sets $D(\Gamma)$, and $D^i(\Gamma)$, $i \in N$ are all closed in $X(\Gamma)$ and compact.

Proof: Apparently, $D(\Gamma)$ will be closed and compact if for each i , $i \in N$, the set $D^i(\Gamma)$ is closed and compact because the set $D(\Gamma)$ consists of the intersection of the sets $D^i(\Gamma)$, $\forall i \in N$.

For any player i , $i \in N$, if $D^i(\Gamma) = SD^i(\Gamma)$ then, and by Lemma 4.1, $D^i(\Gamma)$ is closed in $X(\Gamma)$ and compact. The same holds true if for some i , $i \in N$, $D^i(\Gamma) = X(\Gamma)$. Suppose then that for some, arbitrary, player i , $i \in N$, $D^i(\Gamma) \neq X(\Gamma)$, and therefore, $(X(\Gamma) - D^i(\Gamma)) \neq \emptyset$. Let $x, x \in (X(\Gamma) - D^i(\Gamma))$, be arbitrary. There are two distinct possibilities for any payoff vector x , as above, either (a) $x \notin \bar{F}(\Gamma)$, or (b) $x \in \bar{F}(\Gamma)$.

Case (a): $x \notin \bar{F}(\Gamma)$.

Note that both sets $SD^i(\Gamma)$ and $\bar{F}(\Gamma)$ are closed in $X(\Gamma)$. Therefore, r , such that

$$(4.42) \quad r \equiv \min\{d(x, \bar{F}(\Gamma)), d(x, SD^i(\Gamma))\},$$

is well defined, and greater than zero. Furthermore, $x \notin \bar{F}(\Gamma)$ and $x \notin D^i(\Gamma)$, imply that the set $Nd_r(x)$ does not intersect the set $D^i(\Gamma)$. Therefore, $Nd_r(x) \subseteq (X(\Gamma) - D^i(\Gamma))$.

Case (b): $x \in \bar{F}(\Gamma)$.

Note that $x \in \bar{F}(\Gamma)$, and $x \notin D^1(\Gamma)$ imply that there exists an i.r.p.c.

$(\tilde{x}; \tilde{\tau}_1) \in EB^1_\Gamma(x)$, such that, if $\tilde{\delta}$ is the real number defined by

$$(4.43) \quad \tilde{\delta} \equiv \max_{\hat{c} \in C^1_N} \{v(\hat{c}) - \tilde{x}(\hat{c} - \{i\})\},$$

then $x^i > \tilde{\delta}$. Apparently, $\tilde{\delta}$ is the maximum payoff that player i can get

as a result of any "counter action" that he can take against the action

$(\tilde{x}; \tilde{\tau}_1)$. But, $(x^i - \tilde{\delta} > 0)$ implies that there exists a positive real

number ε such that if $\bar{x} \in X(\Gamma)$, and if $|\bar{x}^j - x^j| < \varepsilon \forall j \in N$, then, either

$(\bar{x}; \tilde{\tau}_1) \in EB^1_\Gamma(\bar{x})$ or, if $(\bar{x}; \tilde{\tau}_1) \notin EB^1_\Gamma(\bar{x})$, there exists a redistribution of

the value of each non-trivial coalition $\tilde{c}$, $\tilde{c} \in \tilde{\tau}_1$, yielding a new payoff

vector, say $\tilde{x}^*$, such that the i.r.p.c. $(\tilde{x}^*; \tilde{\tau}_1)$ will satisfy conditions

(ii) and (iii) in part (a.2) of Definition 4.5, and $(\tilde{x}^*; \tilde{\tau}_1) \in EB^1_\Gamma(\bar{x})$. (*)

But then, we can find r , $r > 0$, such that $Nd_r(x) \subseteq (X(\Gamma) - D^1(\Gamma))$.

q.e.d.

4.5 The Effective Core

Following the two-step process underlined in the introduction of this chapter, a new solution concept can be defined out of the Pareto optimal points of the set of individually defensible positions of a game Γ . For reasons that we will explain below, we call this solution concept the effective core of a game.

Definition 4.8: Let $\Gamma = (N, v)$. The set $EK(\Gamma)$, $EK(\Gamma) \equiv (X^*(\Gamma) \cap D(\Gamma))$, constitutes the effective core of the game Γ .

(*) Note that conditions (ii) and (iii) of Definition 4.5 are distribution requirements for each coalition $\tilde{c} \in \tilde{\tau}_1$. Hence, $\tilde{x}$ may violate these requirements with respect to $\tilde{x}$, and a redistribution may be necessary.

Using Theorem 4.4, we can restate the definition of the set $EK(\Gamma)$ as follows

$$(4.44) \quad EK(\Gamma) \equiv \{x \in X(\Gamma): x \in X^*(\Gamma), \text{ and } EB_{\Gamma}(x) = \emptyset\}$$

In words, the effective core of a game Γ consists of all Pareto optimal payoff vectors of Γ that are not effectively dominated.

Due to the interpretation given in Section 4.4 to the set of individually defensible positions of a game, for any payoff vector x , $x \in D(\Gamma)$, we could say that every player i , $i \in N$, is effective for x , given the rules of the game. In particular, and were we to apply the notions of α -effectiveness and β -effectiveness, introduced by Aumann and Peleg [5], and Aumann [1], not to coalitions in general, but to individual players, we could give the following interpretation to the elements of the sets $SD(\Gamma)$ and $D(\Gamma)$. A payoff vector x belongs to the set $SD(\Gamma)$ if every player i , $i \in N$, can assure himself of his payoff x^i independently of the actions of $N - \{i\}$. On the other hand, a payoff vector x belongs to the set $D(\Gamma)$ if, for each player i , $i \in N$, the set of players $N - \{i\}$ cannot prevent player i from obtaining his payoff x^i . Apparently, these two statements about individual effectiveness paraphrase the statements about α -effectiveness and β -effectiveness, respectively, in [5].

Now, in Theorem 4.3, Section 4.3, individual effectiveness in the sense of the sets $SD(\Gamma)$ and Pareto optimality were shown to define the core of a game. In a similar way then, individual effectiveness in the sense of the sets $D(\Gamma)$ and Pareto optimality can be utilized to define a

generalized concept of the core, which we have called the effective core of a game. (*)

Apparently, the core of a game Γ is a subset of the effective core of that game because the set $SD(\Gamma)$ is a subset of the set $D(\Gamma)$ and by Theorem 4.3 the core consists of the set of individually strongly defensible positions of Γ which are Pareto optimal.

What may not be so apparent is the following. The effective core of a game may contain more payoff vectors than the core, even if the latter is not empty. To show that this, indeed, may be the case let us consider the following 5-person game. (**)

$N = \{1, 2, 3, 4, 5\}$; and

$v(c) = \min\{\#(c \cap P), \frac{1}{2} \#(c \cap Q)\}$, $\forall c, c \in C_N, c \neq \emptyset$, where:

$P = \{1, 2\}$, $Q = \{3, 4, 5\}$, and $\#S$ denotes the number of elements in the set S .

Because the players in each set P and Q are symmetric, let

$P = \{i, j\}$, and $Q = \{p, q, r\}$. The set of Pareto optimal payoff vectors of the game consists of all payoff vectors $x^*, x^* \in X(\Gamma)$, such that

(*) Since individual effectiveness in the sense of the sets $SD(\Gamma)$ and Pareto optimality yields the usual core or the α -core of a game, [1], [5], one may ask the following interesting question, to which we do not have the answer at the present. How is the effective core related to the β -core?

(**) Games like this have been used by Maschler [20] in defense of the argument that the bargaining set $M_1^{(i)}$ may contain payoff vectors which are more plausible, as an outcome of the game, than payoff vectors in the core.

$x^*(N) = 1.5$. They are attainable through coalition structures τ^* of the form $\tau^* = \{\{i, p\}, \{j, q, r\}\}$, i.e., the coalition structures τ^* consist of two coalitions. The first coalition contains one player from the set P and one player from the set Q, while the second coalition contains the remaining three players of N.

The core of the game (see [20]) consists of the single point $x = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which, of course, belongs to the effective core of the game.

Now, let us consider the payoff vector $\bar{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This payoff vector gives to any coalition $c = \{i, p\}$ more than its value. Hence, it is dominated by the coalition $\tilde{c} = N - c$, i.e., $\tilde{c} = \{j, q, r\}$, which is the only coalition that dominates $\bar{x}$, given c . The i.r.p.c. $(\tilde{x}; \tilde{\tau})$ that satisfy conditions (i) - (iii), in part (a.2) of Definition 4.5, for some player, are of the form

$$\tilde{\tau} = \{\tilde{c}, \{i\}, \{p\}\}, \text{ and}$$

$$\tilde{x} \in \{x \in X(\Gamma) : x^i = 0, x^p = 0, (x^j + x^q + x^r) = 1, (x^q + x^r) \leq \frac{3}{4}, \frac{1}{4} \leq x^j \leq \frac{1}{3}, \frac{1}{3} \leq x^q \leq \frac{5}{12}, \text{ and } \frac{1}{3} \leq x^r \leq \frac{5}{12}\}.$$

But if $\hat{c} = \{i, q, r\}$, then $1 = v(\hat{c}) \geq (\tilde{x}(\{q, r\}) + \tilde{x}^i)$. Hence, $\bar{x}$ is a defensible position for player i. On the other hand, if $\hat{c} = \{p, s, j\}$, where player s is the player in $\{q, r\}$ such that: $\tilde{x}^s = \min\{x^q, x^r\}$, then, $1 = v(\hat{c}) \geq (\tilde{x}(\{j, s\}) + \tilde{x}^p)$. Hence $\bar{x}$ is a defensible position for player p. By symmetry of the players in each set P and Q, $\bar{x}$ is an individually defensible position for the game.

For the proof of Theorem 4.5, below, we need the following lemma which states that, for a game Γ with an empty core, each payoff vector x , $x \in X(\Gamma)$, is a defensible position for at least one of the players.

Lemma 4.3: Let $\Gamma = (N, v)$ be such that $K(\Gamma) = \emptyset$. Then,

$$(x \in X(\Gamma)) \Rightarrow (x \in \bigcup_{i \in N} D^i(\Gamma)).$$

Proof: Let x be an arbitrary payoff vector in the set $X(\Gamma)$, and suppose that $x \notin \bigcup_{i \in N} D^i(\Gamma)$. Then, $x \notin \bigcup_{i \in N} SD^i(\Gamma)$. Therefore, and from the definition of the set $F(\Gamma)$, i.e., $F(\Gamma) = (X(\Gamma) - \bigcup_{i \in N} SD^i(\Gamma))$, it will follow that $x \in F(\Gamma)$.

For each player i , $i \in N$, let us define the function

$$\Delta^i: X(\Gamma) \rightarrow E^1 \text{ by}$$

$$(4.45) \quad \Delta^i(x) \equiv x^i - v(\{i\}),$$

i.e., for any payoff vector x , $\Delta^i(x)$ is the difference between what player i receives at x and what he can get on his own.

From Theorem 4.1, and the definition of the set $F(\Gamma)$,

$$(4.46) \quad (x \in F(\Gamma)) \Rightarrow (\Delta^i(x) > 0, \forall i \in N).$$

On the other hand, that $x \in F(\Gamma)$ and $x \notin \bigcup_{i \in N} D^i(\Gamma)$, implies that for each player i , $i \in N$, there exists an i.r.p.c. $(\tilde{x}_i; \tilde{\tau}_i)$, $(\tilde{x}_i; \tilde{\tau}_i) \in EB_\Gamma^i(x)$, such that

$$(4.47) \quad x_i(c_i - \{i\}) + p^i(x) > v(c_i), \text{ for each } c_i \in C_N^i.$$

Note that the subscript i in the i.r.p.c. $(\tilde{x}_i; \tilde{\tau}_i)$ indicates that $(\tilde{x}_i; \tilde{\tau}_i) \in EB_{\Gamma}^1(x)$. Likewise, the subscript i is used in c_i to indicate that coalition c_i contains player i . Using the projection function $p^i(\cdot)$ to denote the i -th element x^i of the vector x , and not the notation followed so far, will help us avoid some confusion below. Recall that $(\tilde{x}_i; \tilde{\tau}_i) \in EB_{\Gamma}^1(x)$ implies that $\{i\} \in \tilde{\tau}_i$, for any $i \in N$. Therefore, and because $(\tilde{x}_i; \tilde{\tau}_i)$ is an i.r.p.c., then, $p^i(\tilde{x}_i) = v(\{i\})$, and we can rewrite (4.47) as

$$\tilde{x}_i(c_i) + \Delta^i(x) > v(c_i), \text{ for each } c_i \in C_N^i,$$

or

$$(4.48) \quad \tilde{x}_i(c_i) > v(c_i) - \Delta^i(x), \text{ for each } c_i \in C_N^i.$$

Now, for any player r , $r \in N$, $(\tilde{x}_r; \tilde{\tau}_r) \in EB_{\Gamma}^r(x)$ implies that there exists at least one coalition $\tilde{c}$, $\tilde{c} \in \tilde{\tau}_r$, such that: $\tilde{x}_r \underline{\text{dom}}_{\tilde{c}} x$. The set S , defined below, contains all players that could possibly belong to such coalitions, for any $r \in N$, i.e.,

$$(4.49) \quad S \equiv \{j \in N: (C_N^j \cap C_{\Gamma}(\tilde{x}_r d x) \cap \tilde{\tau}_r) \neq \emptyset, \forall r \in N\}$$

For any player i , $i \in S$, the set $\bar{c}^i$, defined below contains all players r , $r \in N$, such that for a coalition $\tilde{c}$ that contains player i , $\tilde{x}_r \underline{\text{dom}}_{\tilde{c}} x$, i.e.,

$$(4.50) \quad \bar{c}^i \equiv \{r \in N: (C_N^i \cap C_{\Gamma}(\tilde{x}_r d x) \cap \tilde{\tau}_r) \neq \emptyset\}, \text{ any } i \in S.$$

Let us observe that for any player i , $i \in S$, and for any player r , $r \in \bar{c}^i$, the set $(C_N^i \cap C_{\Gamma}(\tilde{x}_r d x) \cap \tilde{\tau}_r)$ contains a single element. This

follows from the fact that $\tilde{\tau}_r$ is a coalition structure. Let us denote this element by $\tilde{c}_1^r$, i.e.,

$$(4.51) \quad \tilde{c}_1^r = (C_N^1 \cap C_r(\tilde{x}_r, d, x) \cap \tilde{\tau}_r), \text{ for each } r \in \tilde{c}^1.$$

But then, for each player i , $i \in S$, $(\tilde{x}_i; \tilde{\tau}_i) \in EB^1(x)$ implies that (4.48) must hold for every coalition $\tilde{c}_1^r$, $r \in \tilde{c}^1$. Therefore,

$$(4.52) \quad \tilde{x}_i(\tilde{c}_1^r) > v(\tilde{c}_1^r) - \Delta^1(x), \text{ for each } r \in \tilde{c}^1.$$

But, $\tilde{c}_1^r \in \tilde{\tau}_r$, and $(\tilde{x}_r; \tilde{\tau}_r)$ an i.r.p.c., imply that $\tilde{x}_r(\tilde{c}_1^r) = v(\tilde{c}_1^r)$.

It will follow then that

$$(4.53) \quad \tilde{x}_i(\tilde{c}_1^r) > \tilde{x}_r(\tilde{c}_1^r) - \Delta^1(x), \text{ for each } r \in \tilde{c}^1.$$

Summing over all players $r \in \tilde{c}^1$, we obtain

$$(4.54) \quad \sum_{r \in \tilde{c}^1} \tilde{x}_i(\tilde{c}_1^r) > \sum_{r \in \tilde{c}^1} \tilde{x}_r(\tilde{c}_1^r) - \# \tilde{c}^1 \Delta^1(x)$$

where $\# \tilde{c}^1$ denotes the number of players in the set $\tilde{c}^1$. Summing over all players $i \in S$, we obtain

$$(4.55) \quad \sum_{i \in S} \sum_{r \in \tilde{c}^1} \tilde{x}_i(\tilde{c}_1^r) > \sum_{i \in S} \sum_{r \in \tilde{c}^1} \tilde{x}_r(\tilde{c}_1^r) - \sum_{i \in S} \# \tilde{c}^1 \Delta^1(x),$$

or, and by using the definition of $\tilde{x}_r(\tilde{c}_1^r)$, and $\tilde{x}_i(\tilde{c}_1^r)$,

$$(4.56) \quad \sum_{i \in S} \sum_{r \in \tilde{c}^1} \sum_{j \in \tilde{c}_1^r} p^j(\tilde{x}_i) > \sum_{i \in S} \sum_{r \in \tilde{c}^1} \sum_{j \in \tilde{c}_1^r} p^j(\tilde{x}_r) - \sum_{i \in S} \# \tilde{c}^1 \Delta^1(x).$$

Let,

$$(4.57) \quad \tilde{c}_i^{r+} \equiv \{j \in \tilde{c}_i^r : p^j(\tilde{x}_i) \geq p^j(\tilde{x}_r)\}, \text{ and}$$

$$\tilde{c}_i^{r-} \equiv \tilde{c}_i^r - \tilde{c}_i^{r+}.$$

In words, $\tilde{c}_i^{r+}$ denotes the set of players in coalition $\tilde{c}_i^r$ who receive in $\tilde{x}_i$ a payoff at least as great as in $\tilde{x}_r$, while $\tilde{c}_i^{r-}$ denotes the players in $\tilde{c}_i^r$ who receive in $\tilde{x}_i$ a payoff smaller than what they receive in $\tilde{x}_r$. So we can rewrite (4.56) as

$$(4.58) \quad \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r+}} p^j(\tilde{x}_i) + \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r-}} p^j(\tilde{x}_i) \\ > \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r-}} p^j(\tilde{x}_r) - \sum_{i \in S} \# \tilde{c}_i^+ \Delta^i(x).$$

For each player i , $i \in N$, let s^i be the set defined by

$$(4.59) \quad s^i \equiv \{j \in N : (C_N^j \cap C_r(\tilde{x}_i, dx) \cap \tilde{c}_i^+) \neq \emptyset\}.$$

(Note that $S = \bigcup_{i \in N} s^i$.) Then,

$$(4.60) \quad \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r-}} p^j(\tilde{x}_r) \equiv \sum_{i \in N} \sum_{j \in s^i} p^j(\tilde{x}_i).$$

But each player j , $j \in \tilde{c}_i^{r+}$, is such that $j \in s^r$. Hence,

$$(4.61) \quad \sum_{i \in N} \sum_{j \in s^i} p^j(\tilde{x}_i) - \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r+}} p^j(\tilde{x}_i) \\ = \sum_{i \in S} \sum_{r \in \tilde{c}_i^+} \sum_{j \in \tilde{c}_i^{r-}} p^j(\tilde{x}_r),$$

and (4.60) in conjunction with (4.61) will imply that

$$\begin{aligned}
 (4.62) \quad & \sum_{i \in S} \sum_{r \in \bar{c}_i^{-1}} \sum_{j \in \bar{c}_i^{r-}} p^j(\tilde{x}_r) - \sum_{i \in S} \sum_{r \in \bar{c}_i^{-1}} \sum_{j \in \bar{c}_i^{r+}} p^j(\tilde{x}_i) \\
 & = \sum_{i \in S} \sum_{r \in \bar{c}_i^{-1}} \sum_{j \in \bar{c}_i^{r-}} p^j(\tilde{x}_r).
 \end{aligned}$$

It will follow then that by using (4.62), we can reduce (4.58) to

$$(4.63) \quad \sum_{i \in S} \# \bar{c}_i^{-1} \Delta^i(x) > \sum_{i \in S} \sum_{r \in \bar{c}_i^{-1}} \sum_{j \in \bar{c}_i^{r-}} (p^j(\tilde{x}_r) - p^j(\tilde{x}_i)).$$

Observe now that for any player i , $i \in N$, $p^i(\tilde{x}_i) = v(\{i\})$. So, $i \in \bar{c}_i^{r-}$.

On the other hand, $\tilde{x}_r \in \text{dom}_{\bar{c}_i^{-1}} x$. Hence, $p^i(\tilde{x}_r) \geq p^i(x)$. But then, $p^i(\tilde{x}_r) - p^i(\tilde{x}_i) = p^i(\tilde{x}_r) - v(\{i\}) \geq \Delta^i(x)$. Therefore, (4.63) is reduced to

$$(4.64) \quad 0 > \sum_{i \in S} \sum_{r \in \bar{c}_i^{-1}} \sum_{\substack{j \in \bar{c}_i^{r-} \\ j \neq i}} (p^j(\tilde{x}_r) - p^j(\tilde{x}_i)),$$

which is a contradiction, because for every player j , $j \in \bar{c}_i^{r-}$, $p^j(\tilde{x}_r) - p^j(\tilde{x}_i) > 0$ by the definition of the set $\bar{c}_i^{r-}$ in (4.57).

q.e.d.

Theorem 4.5, below states that every game $\Gamma = (N, v)$ has a non-empty effective core.

Theorem 4.5: Let $\Gamma = (N, v)$. Then, $EK(\Gamma) \neq \emptyset$.

Proof: Apparently, if the core of Γ is not empty then $K(\Gamma) \subseteq EK(\Gamma)$

will imply that $EK(\Gamma) \neq \emptyset$. Assume then that $K(\Gamma) = \emptyset$.

Recall that, by Definition 4.8, $EK(\Gamma) = (X^*(\Gamma) \cap D(\Gamma))$, while, by Definition 4.6, $D(\Gamma) = \bigcap_{i \in N} D^i(\Gamma)$. For each player i , $i \in N$, let us define the set $D^{*i}(\Gamma)$ by

$$(4.65) \quad D^{*i}(\Gamma) \equiv D^i(\Gamma) \cap X^*(\Gamma),$$

i.e., the set $D^{*i}(\Gamma)$ consists of the Pareto optimal points of the set $D^i(\Gamma)$. It will follow then that

$$(4.66) \quad EK(\Gamma) = \bigcap_{i \in N} D^{*i}(\Gamma).$$

Therefore, the proof of the theorem will be completed if we show that

$$\bigcap_{i \in N} D^{*i}(\Gamma) \neq \emptyset.$$

Before we proceed to do that, let us observe the following.

- (a) For each player i , $i \in N$, $D^{*i}(\Gamma) \neq \emptyset$, i.e., any x , $x \in X^*(\Gamma)$, such that $x^i = v(\{i\})$, belongs to the set $D^i(\Gamma)$. (b) For each player i , $i \in N$, the set $D^{*i}(\Gamma)$ is a compact set, which follows from Lemma 4.2 in conjunction with the fact that $X^*(\Gamma)$ is a compact set. I.e., by (4.65), $D^{*i}(\Gamma)$ consists of the intersection of two compact sets. (c) The set $X^*(\Gamma)$, consisting of the Pareto optimal payoff vectors of $X(\Gamma)$ is a convex and compact set.

With the necessary modifications we prove that $\bigcap_{i \in N} D^{*i}(\Gamma) \neq \emptyset$, following Peleg's method about the existence of the bargaining set $M_1^{(i)}$ in [27].

For any payoff vector x , $x \in X^*(\Gamma)$, the compactness of each set $D^{*i}(\Gamma)$ implies that

$$(4.67) \quad d(x, D^{*i}(\Gamma)) = \underset{\bar{x} \in D^{*i}(\Gamma)}{\text{Minimum}} \{d(x, \bar{x})\}.$$

Apparently, $(x \in D^{*i}(\Gamma)) \Rightarrow d(x, D^{*i}(\Gamma)) = 0$, for any player i , $i \in N$. Hence, to show that $\bigcap_{i \in N} D^{*i}(\Gamma) \neq \emptyset$, it will be sufficient to show that there exists a payoff vector $\hat{x}$, $\hat{x} \in X^*(\Gamma)$, such that $d(\hat{x}, D^{*i}(\Gamma)) = 0$, $\forall i \in N$. Suppose to the contrary that no such payoff vector exists in $X^*(\Gamma)$. Then, for each x , $x \in X^*(\Gamma)$, there exists a player j , $j \in N$, such that $d(x, D^{*j}(\Gamma)) > 0$.

For each player i , $i \in N$, let $g^i(\cdot)$, $g^i: X^*(\Gamma) \rightarrow E^1$, be the function defined by

$$(4.68) \quad g^i(x) \equiv \Delta^i(x) - d(x, D^{*i}(\Gamma)),$$

where the function $\Delta^i(x)$ has been already defined in (4.45). Note that $\Delta^i(x) \geq 0$, and that $d(x, D^{*i}(\Gamma)) \leq \Delta^i(x)$. Hence, $g^i(x) \geq 0$, $\forall x \in X^*(\Gamma)$, and $\forall i \in N$. Let,

$$(4.69) \quad \delta \equiv \underset{x \in X^*(\Gamma)}{\text{Min}} \underset{i \in N}{\text{Max}} \{d(x, D^{*i}(\Gamma))\}.$$

If the hypothesis that for each x , $x \in X^*(\Gamma)$, there exists a player j , $j \in N$, such that $d(x, D^{*j}(\Gamma)) > 0$ is correct, then $\delta > 0$. Therefore, if for each player i , $i \in N$, we define the function $h^i(\cdot)$, $h^i: X^*(\Gamma) \rightarrow E^1$, by

$$(4.70) \quad h^i(x) \equiv \frac{g^i(x) + \frac{\delta}{2} + |\Delta^i(x) - g^i(x) - \frac{\delta}{2}|}{\sum_{r \in N} g^r(x) + n \frac{\delta}{2} + \sum_{r \in N} |\Delta^r(x) - g^r(x) - \frac{\delta}{2}|},$$

we must have that $h^i(x) > 0$, and $\sum_{i \in N} h^i(x) = 1$.

For each player i , $i \in N$, let us consider now the function $f^i(\cdot)$, $f^i: X^*(\Gamma) \rightarrow E^1$, defined by

$$(4.71) \quad f^i(x) \equiv v(\{i\}) + h^i(x) \sum_{r \in N} \Delta^r(x).$$

Observe that $\sum_{i \in N} f^i(x) = \sum_{i \in N} v(\{i\}) + \sum_{r \in N} \Delta^r(x)$, because

$$\sum_{i \in N} h^i(x) = 1. \quad \text{But} \quad \sum_{r \in N} \Delta^r(x) = \sum_{r \in N} x^r - \sum_{r \in N} v(\{r\}).$$

Therefore, $\sum_{i \in N} f^i(x) = x(N)$. On the other hand, $v(\{i\}) \leq f^i(x) \leq x(N)$, because $0 < h^i(x) \leq 1$, for each $i \in N$. As a consequence, the vector valued function $f(\cdot)$ defined by

$$(4.72) \quad f(x) \equiv (f^1(x), \dots, f^n(x))$$

is such that $f: X^*(\Gamma) \rightarrow X^*(\Gamma)$. In addition, because each function $f^i(\cdot)$ is continuous over $X^*(\Gamma)$, then, $f(\cdot)$ is continuous over $X^*(\Gamma)$.

Furthermore, $X^*(\Gamma)$ is a convex and compact set. Therefore, by Brower's fixed point theorem, there exists a payoff vector x^* , $x^* \in X^*(\Gamma)$, such that $f(x^*) = x^*$.

Given the payoff vector x^* , as above, from among all players r , $r \in N$, for whom, according to the standing hypothesis, $d(x^*, D^{*r}(\Gamma)) > 0$, let player j be the one for whom $d(x^*, D^{*j}(\Gamma)) \geq d(x^*, D^{*r}(\Gamma))$. Then, and by (4.69), $d(x^*, D^{*j}(\Gamma)) \geq \delta$. It will follow then, from (4.68), that

$$(4.73) \quad \Delta^j(x^*) - g^j(x^*) = d(x^*, D^{*j}(\Gamma)) \geq \delta.$$

Hence,

$$(4.74) \quad |\Delta^j(x^*) - g^j(x^*) - \frac{\delta}{2}| = \Delta^j(x^*) - g^j(x^*) - \frac{\delta}{2}.$$

Applying (4.74) to (4.70), we obtain that with respect to player j as above, and with respect to the payoff vector x^* ,

$$(4.75) \quad h^j(x^*) = \frac{\Delta^j(x^*)}{\sum_{r \in N} g^r(x^*) + n \frac{\delta}{2} + \sum_{r \in N} |\Delta^r(x^*) - g^r(x^*) - \frac{\delta}{2}|}$$

But, $f(x^*) = x^*$. So, and by (4.71),

$$(4.76) \quad x^{*j} = v(\{j\}) + \frac{\Delta^j(x^*)}{\sum_{r \in N} g^r(x^*) + n \frac{\delta}{2} + \sum_{r \in N} |\Delta^r(x^*) - g^r(x^*) - \frac{\delta}{2}|} \sum_{r \in N} \Delta^r(x^*).$$

By Lemma 4.3, if $K(\Gamma) = \emptyset$, then, $(x \in X(\Gamma)) \Rightarrow (x \in \bigcap_{i \in N} D^i(\Gamma))$, and one of the standing hypotheses here is that $K(\Gamma) = \emptyset$. Therefore, there exists a player s , $s \in N$, such that $x^* \in D^s(\Gamma)$. This means that $\Delta^s(x^*) - g^s(x^*) = d(x^*, D^s(\Gamma)) = 0$. In turn, this implies that

$$(4.77) \quad |\Delta^s(x^*) - g^s(x^*) - \frac{\delta}{2}| > (\Delta^s(x^*) - g^s(x^*) - \frac{\delta}{2}).$$

Hence,

$$(4.78) \quad \sum_{r \in N} |\Delta^r(x^*) - g^r(x^*) - \frac{\delta}{2}| > \sum_{r \in N} (\Delta^r(x^*) - g^r(x^*) - \frac{\delta}{2})$$

Applying (4.78) to (4.76), and performing the operations of addition, we obtain that

$$(4.79) \quad x^{*j} < v(\{j\}) + \frac{\Delta^j(x^*)}{\sum_{r \in N} \Delta^r(x^*)} \sum_{r \in N} \Delta^r(x^*) = v(\{j\}) + \Delta^j(x^*)$$

or, and in view of the definition of $\Delta^j(\cdot)$ in (4.45), that

$x^{*j} < x^{*j}$, a contradiction.

4.6 The Bargaining Set $M_1^{(i)}$, the Kernel, and Individually Defensible Positions for a Game

The theory developed in this chapter, is, in part, closely related to the "bargaining theory" first introduced by Aumann and Maschler [4]. As we have pointed out in the introduction of this chapter, both theories start out from the common hypothesis that a "security criterion" is sufficient to induce stability of payoff vectors in a cooperative game. Thus, the question is how individually defensible positions in a game are related to the various bargaining sets of Aumann and Maschler.

For a given game $\Gamma = (N, v)$ we make the comparisons relative to the Bargaining Set $M_1^{(i)}$ of that game. (For a precise definition of this particular set see Davis and Maschler [8], or Peleg [27].)

Given a cooperative game $\Gamma = (N, v)$, and for any non-empty coalition $c \in C_N$, let us denote by $(y^i)_{i \in c}$ a distribution of the value of coalition c , $v(c)$, among its members, i.e., $\sum_{i \in c} y^i = v(c)$.

For any i.r.p.c. $(x; \tau)$, $(x; \tau) \in PC(\Gamma)$, let $\tilde{c} \in \tau$, and let $r, s \in \tilde{c}$ be any two distinct players in coalition $\tilde{c}$. Given a coalition $\bar{c}, \bar{c} \in C_N^r$, $\bar{c} \notin C_N^s$, and given a distribution $(\bar{y}^i)_{i \in \bar{c}}$ of the value of that coalition among its members, the pair $((\bar{y}^i)_{i \in \bar{c}}; \bar{c})$ is called an objection of player r against player s in $(x; \tau)$ if $\bar{y}^r > x^r$, and $\bar{y}^i \geq x^i, \forall i \in \bar{c}$.

Given an i.r.p.c. $(x; \tau)$, $(x; \tau) \in PC(\Gamma)$, let $((\bar{y}^i)_{i \in \bar{c}}; \bar{c})$ be an objection of player r against player s in $(x; \tau)$, and let the pair $((\hat{y}^i)_{i \in \hat{c}}; \hat{c})$ be such that $\hat{c} \in C_N^s$, $\hat{c} \notin C_N^r$, and $(\hat{y}^i)_{i \in \hat{c}}$ be a distribution of the value of coalition $\hat{c}$. The pair $((\hat{y}^i)_{i \in \hat{c}}; \hat{c})$, as above, is called a counter objection to the above objection if

- (a) $\hat{y}^i \geq x^i, \forall i \in \hat{c}$, and (b) $\hat{y}^i \geq \bar{y}^i, \forall i \in \bar{c} \cap \hat{c}$.

The Bargaining Set $M_1^{(i)}$ of the game Γ , consists of all i.r.p.c. $(x; \tau) \in PC(\Gamma)$ such that for each objection there exists a counter objection.

Apparently, and in defining the set of individually defensible positions of a game Γ , we could interpret the various "actions" and "counter actions," introduced in this work, as constituting some kind of "objections" and "counter objections." However, this will not fit into any of the categories of objections and counter objections introduced in [4], or the ones defined above. Therefore, there are some distinct differences between the bargaining theory of Aumann and Maschler and the theory developed in this chapter.

First, Aumann and Maschler [4] deal with the problem of stability of i.r.p.c., i.e., the players in a game Γ must agree first upon a coalition structure and then decide on the distribution of the value of each coalition in that coalition structure. No restriction of this kind has been imposed in this chapter. (*)

Second, the emphasis, in defining individually defensible positions for a game, has been on the concerted actions that players in the set $N - \{i\}$ could undertake against the i -th player, relative to some payoff vector x . Thus, these actions can be interpreted as much stronger "objections" than the ones used to define the bargaining set $M_1^{(i)}$, above. But then, it seems reasonable, that player i , using a coalition to counteract such actions, should not have to promise to each player in that coalition more, or less, of a payoff, than what that player could have

(*) However, we should note that a way to generalize the bargaining set $M_1^{(i)}$ for the whole set $X(\Gamma)$, or any subset of payoff vectors, has been suggested by Schmeidler [34].

received by taking part in the concerted action against player i . In other words, and unlike the counter objections in the bargaining set $M_1^{(i)}$, in considering whether a payoff vector x is an individually defensible position for some player $i \in N$, that player, in counteracting an action against him, could seek the help of some of the "losers," like himself. He does not have to promise them more than what they get at the new position. In particular, he does not have to promise them a payoff greater or equal to the original one at x . Hence, and in this sense, we could say that a player in counter acting a concerted action against him could form a "counter objection" against a stronger "objection" much easier than the one in the Bargaining Set $M_1^{(i)}$, given above.

An important implication that follows these conclusions is that if we were to use the various counter objections in [8] as counter actions against the concerted actions of the players in $N - \{i\}$, some of these concerted actions could not be countered by player i . Therefore, it will seem that some elements in the various bargaining sets of a given game Γ , will not, necessarily, be individually defensible positions.

In [7], Davis and Maschler have considered a subset of the bargaining set $M_1^{(i)}$ of a game as a separate solution concept to that game. They called such solution concept, the Kernel of a game.

Given a cooperative game $\Gamma = (N, v)$ and following Schmeidler [34], let $(x; \tau)$ be an i.r.p.c. For each pair of distinct players $i, j \in N$, let

$$(4.80) \quad s_{ij}(x) \equiv \max\{v(c) - x(c) : c \in C_N^i, c \notin C_N^j\}.$$

We say that player i outweighs player j if there exists a coalition $\tilde{c}$, $\tilde{c} \in \tau$, such that $i, j \in \tilde{c}$, and $s_{ij}(x) > s_{ji}(x)$. The Kernel of the coalition structure τ , in the i.r.p.c. $(x; \tau)$, is the set of payoff vectors $x \in \bar{X}_\tau(\Gamma)$, such that for each pair of players (i, j) , i does not outweigh j . The Kernel of the game Γ , Kernel (Γ) , consists of the Kernel of all coalition structures $\tau \in T_N$.

We did claim, above, that some elements in the Bargaining Set $M_1^{(i)}$ of a game Γ , will not, necessarily, be individually defensible positions for that game. Therefore, and because the Kernel of a game Γ , Kernel (Γ) , is a subset of the Bargaining Set $M_1^{(i)}$, if we prove that elements of the Kernel (Γ) are not necessarily individually defensible positions, the above claim will be substantiated.

Theorem 4.6: There exists a game $\Gamma = (N, v)$, and there exists an i.r.p.c. $(x; \tau) \in PC(\Gamma)$, such that $x \in \text{Kernel } (\Gamma)$, but $x \notin D(\Gamma)$.

Proof: Let us consider the following 6-person game:

$$N = \{1, 2, 3, 4, 5, 6\};$$

$$v(\{1, 2, 3\}) = 30, v(\{4, 5, 6\}) = 60,$$

$$v(\{2, 3, 4\}) = v(\{5, 6\}) = 42,$$

$$v(\{1, 3, 5, 6\}) = v(\{1, 2, 5, 6\}) = 62, \text{ and}$$

$$v(c) = 0, \text{ otherwise.}$$

Consider the payoff vector $x = (10, 10, 10, 20, 20, 20)$ corresponding to the coalition structure τ , $\tau = \{\{1, 2, 3\}, \{4, 5, 6\}\}$. Then, $(x; \tau)$ is an i.r.p.c. and it is easily checked that x belongs to the kernel of Γ , and therefore to the bargaining set $M_1^{(i)}$. However, x is not an individually defensible position of Γ . If $\tilde{\tau} = \{\{2, 3, 4\}, \{5, 6\}, \{1\}\}$, and

$\tilde{x} = (0, 11, 11, 20, 21, 21)$, then $(\tilde{x}; \tilde{r})$ is an i.r.p.c. that satisfies conditions (i) - (iii), in part (a.2) of Definition 4.5, for player 1. But $\tilde{x}(c - \{1\}) + 10 > v(c)$, $\forall c \in C_N^1$, because $v(\{1, 2, 3\}) - \tilde{x}(\{2, 3\}) = 8$, $v(\{1, 2, 5, 6\}) - \tilde{x}(\{2, 5, 6\}) = v(\{1, 3, 5, 6\}) - \tilde{x}(\{3, 5, 6\}) = 9$, and $v(c) - \tilde{x}(c - \{1\}) = 0$, for all other $c \in C_N^1$. In other words, $10 > v(c) - \tilde{x}(c - \{1\})$, $\forall c \in C_N^1$. Hence, $(\tilde{x}; \tilde{r}) \in EB_T^1(x)$.

q.e.d.

What is interesting about the game in the proof of the above theorem is that the payoff vector $x = (10, 10, 10, 20, 20, 20)$ is Pareto optimal, and it belongs to the Kernel of that game, hence, to the bargaining set $M_1^{(i)}$, as well. But then, the conclusion of the theorem will imply that even Pareto optimal points of the Kernel and the bargaining set $M_1^{(i)}$ of a game Γ may not, necessarily, be individually defensible positions for that game.

In concluding this section we should point out that, in a game $\Gamma = (N, v)$, the payoff vector where every player receives what he can get on his own, i.e., the vector $(v(\{i\}))_{i \in N}$, belongs to all three sets, i.e., the Kernel, the bargaining set $M_1^{(i)}$, and the set of individually defensible positions of that game. Hence, the three have a non-empty intersection. However, whether the Pareto optimal points of the three sets have a non-empty intersection (hence whether the effective core contains any elements of the other two sets) for every game $\Gamma = (N, v)$, is an open question.

CHAPTER FIVE INDIVIDUALLY DEFENSIBLE ALLOCATIONS FOR AN ECONOMY

5.1 Introduction

Transferability of utility is not an essential assumption for the Von Neumann and Morgenstern theory of games, in general, or the theory of the core, in particular. But then, and because the theory developed in the preceding chapter is based, essentially, on the theory of the core, transferability of utility should not be an essential assumption for that theory either. As a consequence, and with the necessary adjustments, one can go ahead and apply all concepts introduced for the first time in Chapter Four to the cooperative games $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, introduced relative to an economy $\underline{e}$ in Chapter Two, by utilizing the notions of dominance introduced in Chapter Three. Thus, and by combining material from the preceding three chapters, we can lay the foundations of a "Theory for Individually Defensible Allocations in an Economy." This is the problem that we shall deal with in this chapter.

In general, given an economic environment, and given our interpretation of the relevant concepts in Chapter Four, to say that an allocation is individually defensible could be interpreted as saying the following. Each individual economic agent can counteract those concerted actions of the others that, potentially, could result in an allocation that affects his level of satisfaction more adversely, and by so doing he will end up at an allocation which will not make him worse off than what he was at the original one. Thus, it is the level of his own satisfaction that each

individual "defends" and not necessarily the particular allocation that we call individually defensible.

At a more technical level, to say that an allocation is individually defensible in a given economy, means the following. Relative to a cooperative game that the set of individuals in that economy might be playing, and relative to some notion of dominance that we can use to define a core-concept for that game, that allocation constitutes an individually defensible position of the cooperative game under consideration. As we have seen in Chapters Two and Three, neither the cooperative game, nor the notion of dominance that can be used in such game, are necessarily unique for a given economy e . Therefore, it is possible that allocations in such an economy can be considered individually defensible in many ways.

Although there may be different ways that allocations in an economy e could be characterized as individually defensible, yet it may be possible to give a general, abstract, definition for such concept.

As may be recalled from Chapter Two, Section 2.7, given an economy e , the cooperative game $\tilde{\Gamma}(e)$ has been defined by restricting the set of joint strategies $\hat{A}^*$ of the more general game $\Gamma(e)$ to the set $\tilde{A}$. On the other hand, the concrete notions of dominance that induce the binary relationships dom^* and $\alpha\text{-dom}$, in Chapter Three, have been obtained from the abstract notion of dominance that induces the abstract binary relationship dom (Definition 3.1). Therefore, and if we were to go ahead and define individually defensible positions for a game $\Gamma(e)$ using the abstract binary relationship dom , then, we will have a general, abstract notion of what it means for an allocation to be individually defensible in an economy e . From there on, concrete notions for that concept can be obtained relative to either the game $\Gamma(e)$ or the game $\tilde{\Gamma}(e)$, and relative to either

the binary relationship $\underline{\text{dom}}^*$ or $\underline{\alpha\text{-dom}}$, or, for that matter, any other concrete binary domination relationship that we could obtain from $\underline{\text{dom}}$, in a straightforward manner. This approach is followed in the next section.

In concluding this introduction we should point out the following. A cooperative game $\Gamma(\underline{e})$, designed to capture conflict of interests in an economy $\underline{e}$ that cooperative games in other forms cannot, may not have the characteristics of a game $\Gamma = (N, v)$. Thus, many of the results obtained in Chapter Four will not, necessarily, be valid for a game $\Gamma(\underline{e})$, or for that matter for a game $\tilde{\Gamma}(\underline{e})$. On the other hand, and because $\Gamma = (N, v)$ is a classical cooperative game in characteristic function form where the effectiveness of each coalition has been predetermined, all the binary domination relationships introduced in Chapter Three relative to the games $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, will be identical, to each other, and to the usual domination relationship introduced in Chapter Four, for a game $\Gamma = (N, v)$. Thus, and although different interpretations may be possible for individually defensible positions in the games $\Gamma(\underline{e})$ and $\tilde{\Gamma}(\underline{e})$, this cannot be the case for a game $\Gamma = (N, v)$, at least in the sense that such concept is used in Chapter Four.

5.2 Individually Defensible Positions

Given an economic environment $\underline{e}$ let us start out with the cooperative game $\Gamma(\underline{e})$. For each final outcome $\bar{y}$, $\bar{y} \in \hat{Y}^*$, and for each individual i , $i \in N$, the set $B_{\Gamma(\underline{e})}^{i-}(\bar{y})$ is defined as follows

$$(5.1) \quad B_{\Gamma(\underline{e})}^{i-}(\bar{y}) \equiv \{\tilde{y} \in B_{\Gamma(\underline{e})}(\bar{y}) : \bar{y} >_1 \tilde{y}\}.$$

In words, given $\bar{y}, \bar{y} \in \hat{Y}^*$, the set $B_{\Gamma(\underline{e})}^{i-}(\bar{y})$ contains all those final outcomes $\tilde{y} \in \hat{Y}^*$ that dominate the final outcome $\bar{y}$ in the sense of Definition 3.1, and make individual i worse off than what he is at $\bar{y}$. Thus, the sets $B_{\Gamma(\underline{e})}^{i-}(\bar{y})$, defined relative to a game $\Gamma(\underline{e})$, are the equivalent to the sets $B_{\Gamma}^{i-}(x)$ which have been defined in (4.7) relative to a game $\Gamma = (N, v)$, after all adjustments for taking care of the non-transferability of utility in $\Gamma(\underline{e})$ have been made.

Expression (5.1) can be restated relative to either one of the concrete binary domination relationships dom^{*} or α -dom, introduced in Chapter Three, using the relevant sets $B_{\Gamma(\underline{e})}^{*}(\bar{y})$, and $B_{\Gamma(\underline{e})}^{\alpha}(\bar{y})$, in (3.4), and (3.10), respectively, to obtain the corresponding sets $B_{\Gamma(\underline{e})}^{*i-}(\bar{y})$ and $B_{\Gamma(\underline{e})}^{\alpha i-}(\bar{y})$. Likewise, all these sets can be defined relative to a game $\tilde{\Gamma}(\underline{e})$.

From now on, restating Definition 4.4, in Chapter Four, relative to either game $\Gamma(\underline{e})$ or $\tilde{\Gamma}(\underline{e})$, and with respect to any of the binary domination relationships dom, dom^{*}, or α -dom, introduced in Chapter Three, is straightforward. We do so, below, with respect to the game $\Gamma(\underline{e})$ and relative to the binary relationship dom.

Definition 5.1: Given a cooperative game $\Gamma(\underline{e})$:

- (a) We say that a final outcome $\bar{y}, \bar{y} \in \hat{Y}^*$, is a strongly defensible position in $\Gamma(\underline{e})$ for player $i, i \in N$, in an abstract sense, if, and only if, $B_{\Gamma(\underline{e})}^{i-}(\bar{y}) = \emptyset$.
- (b) $SD^i(\Gamma(\underline{e})) \equiv \{\bar{y} \in \hat{Y}^* : B_{\Gamma(\underline{e})}^{i-}(\bar{y}) = \emptyset\}$, any $i \in N$.
- (c) $SD(\Gamma(\underline{e})) \equiv \bigcap_{i \in N} SD^i(\Gamma(\underline{e}))$.
- (d) We say that a final outcome $\bar{y}, \bar{y} \in \hat{Y}^*$, is an individually strongly defensible position in $\Gamma(\underline{e})$, in an abstract sense, if, and only if, $\bar{y} \in SD(\Gamma(\underline{e}))$.

Applying the above concepts to the relevant economy $\underline{e}$, we could say the following. A final allocation $\bar{y}$, $\bar{y} \in \hat{Y}^*$, is strongly defensible for individual i , if domination (in an abstract sense, and relative to the game $\Gamma(\underline{e})$) will not make that individual worse off. Likewise, $\bar{y}$, $\bar{y} \in \hat{Y}^*$, is individually strongly defensible in the economy $\underline{e}$, if domination (in an abstract sense, and relative to the game $\Gamma(\underline{e})$) will not make any individual $i \in N$ worse off.

In an obvious manner, all concepts introduced in Definition 5.1, above, can be defined relative to a game $\tilde{\Gamma}(\underline{e})$.

Concrete notions of strongly defensible positions for a player or individually defensible positions for either game $\Gamma(\underline{e})$ or $\tilde{\Gamma}(\underline{e})$ are easy to obtain. In particular, and with respect to the binary relationships dom^{*} and α -dom we can use the sets $B_{\Gamma(\underline{e})}^{*i-}(\bar{y})$ and $B_{\Gamma(\underline{e})}^{\alpha i-}(\bar{y})$, which can be defined as indicated above, to obtain the respective sets $SD^{*i}(\Gamma(\underline{e}))$, $SD^*(\Gamma(\underline{e}))$, or $SD^{\alpha i}(\Gamma(\underline{e}))$, and $SD^{\alpha}(\Gamma(\underline{e}))$, where the superscripts "*" and " α " will indicate the respective concrete binary relationship dom^{*} or α -dom, used for obtaining those sets. The same holds true for a game $\tilde{\Gamma}(\underline{e})$.

To define weakly defensible positions for players in a game $\Gamma(\underline{e})$ let us proceed in a manner similar to that introduced in Section 4.4

Given a game $\Gamma(\underline{e})$, let us denote by $F(\Gamma(\underline{e}))$ the set of final outcomes in the set $\hat{Y}^*$ that do not constitute strongly defensible positions for any player $i \in N$, i.e., let,

$$(5.2) \quad F(\Gamma(\underline{e})) \equiv (\hat{Y}^* - \bigcup_{i \in N} SD^i(\Gamma(\underline{e}))).$$

The closure of the set $F(\Gamma(\underline{e}))$ will be denoted by $\bar{F}(\Gamma(\underline{e}))$, i.e.,

$$(5.3) \quad \bar{F}(\Gamma(\underline{e})) \equiv \{y \in \hat{Y}^*: Nd_r(y) \cap F(\Gamma(\underline{e})) \neq \emptyset, \\ \forall r, r \in E^1, r > 0\}.$$

The set $\bar{F}(\Gamma(\underline{e}))$ then, is the smallest closed set in $\hat{Y}^*$ that contains the set $F(\Gamma(\underline{e}))$.

Given any two alternatives $(\bar{a}; \bar{y})$, $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$, we introduce the following additional notation.

1. If $\tilde{y} \underline{\text{dom}} \bar{y}$ and if $\tilde{c}$ is a coalition in $C(\tilde{a})$ whose existence is guaranteed by Definition 3.1, we shall say that $\tilde{y}$ dominates $\bar{y}$ via coalition $\tilde{c}$, and we shall write

$$(5.4) \quad \tilde{y} \underline{\text{dom}}_{\tilde{c}} \bar{y}.$$

2. The set of all $\tilde{a}$ -admissible coalitions via which $\tilde{y}$ dominates $\bar{y}$, will be denoted by $C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y})$, i.e.,

$$(5.5) \quad C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y}) \equiv \{\tilde{c} \in C(\tilde{a}): \tilde{y} \underline{\text{dom}}_{\tilde{c}} \bar{y}\}.$$

3. All players, other than some player $i \in N$, that do not belong to a coalition in $C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y})$, belong to the set $c^{i-}(\tilde{y} \text{ d } \bar{y})$. Thus,

$$(5.6) \quad c^{i-}(\tilde{y} \text{ d } \bar{y}) \equiv \{s \in (N - \{i\}): s \notin \tilde{c},$$

$$\forall \tilde{c} \in C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y})\}.$$

4. A set of certain distribution possibilities among the players in the set $(N - \{i\})$, some $i \in N$, given the alternatives $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$ is defined as follows

$$\begin{aligned}
 (5.7) \quad H_{\Gamma(\underline{e})}^i(\tilde{y}; \bar{y}) \equiv \{y^* \in \psi(\tilde{a}): & \text{ (a) } y^* \geq_s \tilde{y}, \forall s \in \tilde{c}, \text{ and} \\
 & \forall \tilde{c} \in C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y}); \text{ and (b)} \\
 & ((\exists c^*, c^* \in (C(\tilde{a}) - (C_{\Gamma(\underline{e})}(\tilde{y} \text{ d } \bar{y}) \cup C_N^i)), \\
 & \text{ and } \exists r, r \in (c^* \cap c^{i-}(\tilde{y} \text{ d } \bar{y})), \exists: \\
 & \tilde{y} >_r \bar{y}) \Rightarrow (\exists s, s \in (c^* \cap c^{i-}(\tilde{y} \text{ d } \bar{y})), \\
 & s \neq r, \exists: \bar{y} >_s \tilde{y}, y^* >_s \tilde{y}, \text{ and } y^* \geq_r \bar{y}))\}.
 \end{aligned}$$

Given any two alternatives $(\tilde{a}; \tilde{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, the set $H_{\Gamma(\underline{e})}^i(\tilde{y}; \bar{y})$, some $i \in N$, contains all those allocations y^* attainable through the same joint action $\tilde{a}$ that the allocation $\tilde{y}$ is attainable, which, compared to $\tilde{y}$ and to $\bar{y}$, have certain properties. The first property is that any player who belongs to a coalition $\tilde{c}$, such that $\tilde{y}$ dominates $\bar{y}$ via $\tilde{c}$, should not be made worse off at y^* than what he is at $\tilde{y}$. The second property concerns players that do not belong to such coalitions, other than player i . In particular, and in comparing y^* , $\tilde{y}$, and $\bar{y}$, what holds true for such players is the following. Whenever, (a) there exists an $\tilde{a}$ -admissible coalition c^* such that $\tilde{y}$ does not dominate $\bar{y}$ via coalition c^* , and (b) coalition c^* contains two distinct players, say players r , and s , such that one of them, player r , is made better off at $\tilde{y}$ while the other, player s , is made worse off at $\tilde{y}$, in both cases relative to $\bar{y}$, then, y^* is such that, player s can be made better off at y^* than what he is at $\tilde{y}$, while player r cannot be made worse off at y^* than what he was at $\tilde{y}$.

Let us call the players in the set $c^{i-}(\tilde{y} \text{ d } \bar{y})$ who are better off at $\tilde{y}$ than at $\bar{y}$, "gainers" and the players in the same set who are worse off at $\tilde{y}$ than at $\bar{y}$, "losers." Then, one of the reasons that the set $H_{\Gamma(\underline{e})}^i(\tilde{y}; \bar{y})$ could be empty for some $i \in N$, is the following. Ceteris

paribus, and discounting player i , there does not exist an allocation $y^*, y^* \in \psi(\tilde{a})$, $y^* \neq \tilde{y}$, such that, for each coalition $c^i \in C(\tilde{a})$ that contains both gainers and losers from the set $c^{i-}(\tilde{y} \text{ d } \bar{y})$, the losers in that coalition could improve their position at y^* relative to $\tilde{y}$, while the gainers in the same coalition are not made worse off at y^* relative to the original allocation $\bar{y}$. The implication of this, is that, whenever $\tilde{y} \text{ dom } \bar{y}$, to impose the condition: $H_{\Gamma(e)}^i(\tilde{y}; \bar{y}) = \emptyset$, will be equivalent to imposing certain "distribution requirements" on those coalitions which do not contain player i and none of their members participates in a coalition $\tilde{c}$ such that $\tilde{y} \text{ dom}_{\tilde{c}} \bar{y}$.

It should be noted here that if we were to start from two alternatives $(\tilde{a}; \tilde{y})$ and $(\bar{a}; \bar{y})$, such that, $\tilde{y} \text{ dom } \bar{y}$, and if $\tilde{y}$ fails to satisfy the condition. $H_{\Gamma(e)}^i(\tilde{y} \text{ d } \bar{y}) = \emptyset$, then there may exist an allocation $\hat{y} \in H_{\Gamma(e)}^i(\tilde{y} \text{ d } \bar{y})$ which will satisfy the condition $H_{\Gamma(e)}^i(\hat{y} \text{ d } \bar{y}) = \emptyset$. Therefore, to impose a condition such as: $H_{\Gamma(e)}^i(\tilde{y} \text{ d } \bar{y}) = \emptyset$ in the process of domination for $\bar{y}$ could be equivalent to imposing a restriction on what allocations in $\psi(\tilde{a})$ could be used in such process. Note that if an allocation $\hat{y} \in \psi(\tilde{a})$ exists and satisfies the condition. $H_{\Gamma(e)}^i(\hat{y} \text{ d } \bar{y}) = \emptyset$, then by using $\hat{y}$, in place of $\tilde{y}$, for the domination of $\bar{y}$ in no way will diminish the power of the coalitions in the set $C_{\Gamma(e)}(\tilde{y} \text{ d } \bar{y})$. In fact, what may be true is that. $C_{\Gamma(e)}(\tilde{y} \text{ d } \bar{y}) \subset C_{\Gamma(e)}(\hat{y} \text{ d } \bar{y})$. On the other hand, by using such allocation in place of $\tilde{y}$, we do not require any intercoalitional transfers that could not be present at the allocation $\tilde{y}$. Because both allocations $\tilde{y}$ and $\hat{y}$ are attainable through the same joint action $\tilde{a}$, to use the allocation $\hat{y}$ in place of $\tilde{y}$ for the domination of $\bar{y}$, is one way to say that: " $\hat{y}$ satisfies a more reasonable set of distribution criteria than $\tilde{y}$, given the joint action $\tilde{a}$."

With the notation introduced in (5.2) - (5.7), above, we are in a position to state:

Definition 5.2: Given a cooperative game $\Gamma(\underline{e})$, and given an alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$:

(a) The final outcome $\bar{y}$ is a weakly defensible position in $\Gamma(\underline{e})$ for player i , $i \in N$, in an abstract sense, if, and only if,

(a.1) $\bar{y} \in \bar{F}(\Gamma(\underline{e}))$, and $B_{\Gamma(\underline{e})}^{i-}(\bar{y}) \neq \emptyset$; and

(a.2) for each alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$, such that.

(i) $\tilde{y} \in B_{\Gamma(\underline{e})}^{i-}(\bar{y})$,

(ii) $\{i\} \in \tilde{\tau}_j$, $\forall j \in J$, and

(iii) $H_{\Gamma(\underline{e})}^i(\tilde{y}; \bar{y}) = \emptyset$,

there exists an alternative $(\hat{a}; \hat{y}) \in Z(\Gamma(\underline{e}))$, such that:

(iv) $C_N^i \cap C(\hat{a}) \cap C_{\Gamma(\underline{e})}(\hat{y} \text{ d } \bar{y}) \neq \emptyset$, and

(v) $\hat{y} \succeq_i \bar{y}$.

(b) $WD^i(\Gamma(\underline{e})) \equiv \{\bar{y} \in \hat{Y}^* : \bar{y} \text{ is a weakly defensible position for player } i\}$, $i \in N$.

Several observations are in order with respect to the above definition. The coalition structures $\tilde{\tau}_j$ in part (ii) of (a.2), represent the coalition structures that must be formed with respect to each activity $j \in J$, given the joint action $\tilde{a}$, according to Lemma 2.1, in Chapter Two. Therefore, any alternative $(\tilde{a}; \tilde{y}) \in Z(\Gamma(\underline{e}))$ that satisfies part (ii) of condition (a.2) in the above definition, will be such that in the joint action $\tilde{a}$, player i is "forced" to carry on every activity $j \in J$ on his own. In turn, this constitutes our interpretation of what is meant by: "concerted actions of players in $(N - \{i\})$ against player i ." The distribution requirements for coalitions that belong to the set

$C_{\Gamma(\underline{e})}(\bar{y} \text{ d } \bar{y})$ which were imposed in part (ii), of condition (a.2), in Definition 4.5, Chapter Four, are not necessary here because they will be automatically satisfied in the case of a game with non-transferable utilities (see, e.g., parts (a) and (b) of Definition 3.1, in Chapter Three). Condition (iii) of part (a.2) in Definition 5.2, above, constitutes what we consider to be the equivalent distribution requirements given in the same condition of Definition 4.5, in Chapter Four. Conditions (iv) and (v) in part (a.2) of Definition 5.2, above, are easily seen to correspond to conditions (iv) - (vi) of Definition 4.5, in Chapter Four, adjusted here for the cooperative game $\Gamma(\underline{e})$.

Definition 5.1, in conjunction with Definition 5.2, above yield:

Definition 5.3: Given a cooperative game $\Gamma(\underline{e})$, and given any alternative $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$:

- (a) The final outcome $\bar{y}$ is a defensible position in $\Gamma(\underline{e})$ for player i , $i \in N$, in an abstract sense, if, and only if;

$$\bar{y} \in (SD^i(\Gamma(\underline{e})) \cup WD^i(\Gamma(\underline{e}))).$$
- (b) $D^i(\Gamma(\underline{e})) \equiv (SD^i(\Gamma(\underline{e})) \cup WD^i(\Gamma(\underline{e}))).$
- (c) $D(\Gamma(\underline{e})) \equiv \bigcap_{i \in N} D^i(\Gamma(\underline{e})).$
- (d) The final outcome $\bar{y}$, is an individually defensible position in $\Gamma(\underline{e})$, in an abstract sense, if, and only if, $\bar{y} \in D(\Gamma(\underline{e})).$

This definition is nothing more than the equivalent to Definition 4.6, in Chapter Four, stated here for a game $\Gamma(\underline{e})$ and with respect to final outcomes of such game. Likewise we can restate this definition relative to a game $\tilde{\Gamma}(\underline{e})$, or with respect to the relevant sets that will emerge if we were to use dom^{*} or α -dom in place of the abstract binary

relationship dom. Thus we could have $WD^{*i}(\Gamma(\underline{e}))$, $D^{*i}(\Gamma(\underline{e}))$, $D^*(\Gamma(\underline{e}))$ or $WD^{\alpha i}(\Gamma(\underline{e}))$, $D^{\alpha i}(\Gamma(\underline{e}))$, $D^{\alpha}(\Gamma(\underline{e}))$, and likewise for a game $\tilde{\Gamma}(\underline{e})$, where, again, the superscripts "*" and " α " will indicate whether the binary relationship dom^{*} or α -dom, is used, respectively, in the relevant definitions.

The relevant effective-core-concepts for a game $\Gamma(\underline{e})$ will take the form

$$(5.8) \quad EK(\Gamma(\underline{e})) \equiv \{\bar{y} \in D(\Gamma(\underline{e})): \bar{y} \text{ is Pareto optimal}\},$$

$$(5.9) \quad EK^*(\Gamma(\underline{e})) \equiv \{\bar{y} \in D^*(\Gamma(\underline{e})): \bar{y} \text{ is Pareto optimal}\}, \text{ and}$$

$$(5.10) \quad EK^{\alpha}(\Gamma(\underline{e})) \equiv \{\bar{y} \in D^{\alpha}(\Gamma(\underline{e})): \bar{y} \text{ is Pareto optimal}\},$$

and likewise for a game $\tilde{\Gamma}(\underline{e})$.

In words, $EK(\Gamma(\underline{e}))$ is the abstract effective core, $EK^*(\Gamma(\underline{e}))$ is the effective core^{*}, and $EK^{\alpha}(\Gamma(\underline{e}))$ is the effective α -core of a game $\Gamma(\underline{e})$.

In a cooperative game $\tilde{\Gamma}(\underline{e})$, where, for a given joint action $\tilde{a} \in \tilde{A}$, the set of players N form a common coalition structure, say the coalition structure $\tilde{\tau}$, with respect to each and every activity $j \in J$, the sets in (5.6) and (5.7), above, will take a much simpler form. In particular, the set of players $c^{i-}(\tilde{y} \text{ d } \bar{y})$ will be forming the common coalition structure, consisting of all coalitions in the set. $(\tilde{\tau} - (C_{\tilde{\Gamma}(\underline{e})}^-(\tilde{y} \text{ d } \bar{y}) \cup \{i\}))$. In this case, the condition $H_{\tilde{\Gamma}(\underline{e})}^i(\tilde{y}; \bar{y}) = \emptyset$. could be stated with respect to each coalition c^* that belongs to such coalition structure.

In concluding this section we should point out the following. Given any two alternatives $(\tilde{a}; \tilde{y})$, $(\bar{a}; \bar{y}) \in Z(\Gamma(\underline{e}))$, let us suppose that conditions (a.1), and parts (i) - (iii) of (a.2), in Definition 5.2, are satisfied relative to some player $i \in N$. Then, and relative to $\bar{y}$, an

alternative $(\tilde{a}; \tilde{y})$ that satisfies parts (i) - (iii) of condition (a.2), as above, is our interpretation of what constitutes: "a concerted action of the players in $(N - \{i\})$ that, potentially, could affect player i most adversely." Although those conditions will be reduced to the relevant conditions in Definition 4.5 (Chapter Four) if we were dealing with a game $\Gamma = (N, v)$, still it may be possible that a different set of conditions could lead to the same result. Thus, it may be possible to define weakly defensible positions for a player, in a game $\Gamma(e)$, in a way which will differ from that used in Definition 5.2. What this means is the following. If we were to start from a cooperative game $\Gamma(e)$, and define what we mean by individually defensible positions for that game, then, we can always be definite of how such concept can be applied to a game $\Gamma = (N, v)$. However, if we were to start from a game $\Gamma = (N, v)$, define such concept, and then try to apply it to a game $\Gamma(e)$, as is the case in this work, there is a possibility that we cannot do so in a unique way, even if we were to decide from the outset what notion of dominance to use in such definition. This, of course, is nothing new, because, as we have already pointed out in Section 5.1, the same holds true with respect to the other solution concepts introduced in this study, namely, the various core-concepts in Chapter Three, and it shows some of the limitations of classical cooperative games.

5.3 Crowded Public Goods

In [11], B. Ellickson presented an example of a 3-person economy with a crowded public good to prove that in such cases the following two propositions hold true. (1) The Lindahl equilibrium may not belong to the core. (2) The core may be empty. In obtaining these results in [11], Ellickson used a classical 3-person game in the form $\Gamma = (N, v)$

presented in Chapter Four, above, to represent the relevant economy. Furthermore, the use of a game in this form does not seem to affect the above results. In other words, the two propositions stated above will hold for an economy with crowded public goods, like the one presented in [11], even if we were to use a game without transferable utility.

In the Appendix of this work we demonstrate that for each 3-person classical game $\Gamma = (N, v)$ that has a non-empty core, the effective core of that game is identical to its core. This, in conjunction with the first proposition stated above will imply that the Lindahl equilibrium of an economy with a crowded public good may not belong to the effective core of the relevant game. But the effective core of a game contains all the Pareto optimal points which constitute individually defensible positions for that game. Therefore, in an economy with a crowded public good, the Lindahl equilibrium allocation may not be individually defensible.

Because the effective core of a 3-person game with a non-empty core will contain only individually strongly defensible positions, one may wonder whether the above conclusion is a consequence of such strong criteria for stability, and whether weaker stability criteria could be satisfied by the Lindahl equilibrium. For example, and due to Theorem 4.5 in Chapter Four, the effective core of a 3-person game $\Gamma = (N, v)$ with an empty core, will not be empty, and it will not contain any individually strongly defensible positions. One then may wonder whether the Lindahl equilibrium will satisfy the weaker set of stability criteria utilized in such case. As the example presented below demonstrates this will not be necessarily the case. In particular, and because this example is the same as the one utilized by Ellickson [11, p. 421] to demonstrate the validity of the second proposition mentioned above we can conclude the

following. The Lindahl equilibrium allocation of an economy with a crowded public good may not be individually defensible independently of whether the core of the relevant game $\Gamma = (N, v)$ is empty or not.

For brevity we present here only the relevant game $\Gamma = (N, v)$. (The reader can check that this game represents the case in [11, p. 421] where $\omega_1 = \omega_2 = 4$, $\omega_3 = 3$, $a = b = 1$, and $c = \frac{3}{2}$.)

This game is described as follows:

$$N = \{1, 2, 3\};$$

$$v(\{1\}) = v(\{2\}) = \frac{48}{12}, \quad v(\{3\}) = \frac{27}{12};$$

$$v(\{1, 2\}) = \frac{192}{12}, \quad v(\{1, 3\}) = v(\{2, 3\}) = \frac{147}{12};$$

$$v(N) = \frac{242}{12}.$$

The payoff vector (in units of utility) that corresponds to the Lindahl equilibrium of the 3-person economy represented by the above game is $\bar{x} = (\frac{88}{12}, \frac{88}{12}, \frac{66}{12})$.

In the Appendix we prove a theorem about the payoff vectors that will belong to the effective core of a 3-person game $\Gamma = (N, v)$ with an empty core. From that theorem, in conjunction with the fact that the 3-person game presented above has an empty core, it will follow that, for this particular game

$$EK(\Gamma) = \{x \in X^*(\Gamma) : (\frac{95}{12}, \frac{95}{12}, \frac{50}{12}) \leq x \leq (\frac{96}{12}, \frac{96}{12}, \frac{51}{12})\}.$$

It is easily checked that $\bar{x} \notin EK(\Gamma)$, i.e., the payoff vector $\bar{x}$ that corresponds to the Lindahl equilibrium violates the upper bound for player 3, and the lower bound for both players 1 and 2.

CHAPTER SIX SUMMARY AND CONCLUDING REMARKS

6.1 Summary

A number of questions have been raised in Chapter One and a variety of answers to those questions have been proposed in Chapters Two through Five of this work.

The first major question that we have raised concerns the inadequacy of cooperative games in characteristic function form to capture, and deal with, certain situations of conflict of interest, in particular for economic environments with (a) externalities and public goods, and (b) a multiplicity of economic activities which require the interaction of economic agents from groups with opposing interests.

To deal with this problem, in Chapter Two we have followed a two-step process. First, we have modeled an economic environment in a form where the above conflicts of interest could be clearly identifiable. Second, we have introduced a cooperative game, relative to such an environment, where individual players can use strategies which imply that they can undertake, simultaneously, different economic activities as members of different coalitions.

The basic idea in Chapter Two is to allow the set of economic agents to form different coalition structures with respect to different economic activities in the relevant economy. The major problem that could result from this approach is to identify the strategies of each coalition so formed. In particular, this is so because a coalition could carry on a single activity while its members could carry on

different activities with different coalitions. This problem has been solved by treating coalition actions as conditional strategies. In other words, a coalition formed for the purpose of carrying on some activity, can undertake a coalition action with respect to that activity, but the use of such coalition action as a strategy may be conditional on its members reaching the appropriate agreements with players outside the coalition, with respect to other activities. Thus, the use of such strategies may require the consent of players outside the coalition. The cooperative game that emerges from this setup, is a cooperative game with conditional strategies.

For comparison, the cooperative game that will correspond to the situation where the set of economic agents in an economy can form only a common coalition structure with respect to each and every economic activity, is also presented in Chapter Two. For this game, which closely resembles the cooperative games in characteristic function form used most commonly in the literature, we could say that coalitions can use only independent strategies.

The second major question raised in the Introduction concerns solution concepts to cooperative games in general.

In Chapter Three, and relative to the cooperative games introduced in Chapter Two with respect to a given economic environment, we deal with the problem of defining core-concepts for such games and, in extension, for the respective economies.

The major difficulty in defining core-concepts for a cooperative game with conditional strategies lies with the specification of the effectiveness of each coalition. Although this is also a problem in cooperative games where coalitions can use only independent strategies,

conditionality of strategies makes the problem more complicated. To deal with this problem we have introduced the concept of a consent correspondence which will indicate which ones of its conditional strategies a coalition can use in a given situation, in the sense that players in the complement of that coalition will be willing to go along with coalition actions that involve members of both groups. In a way then, conditional strategies in the consent correspondence of a game can be used as "independent strategies" by the respective coalitions in a given situation.

From this point on, it is easy to define a general notion of dominance, in a game with conditional strategies, which induces an abstract binary domination relationship over the set of final outcomes of that game. Two concrete notions of dominance have been obtained from this general notion of dominance, and the corresponding to them core-concepts, the core^{*} and the α -core, have been introduced in Chapter Three for both, the cooperative game with conditional strategies, and the cooperative game with independent strategies.

Briefly speaking, for the concrete notion of dominance that leads to the core^{*}, alternatives in a cooperative game (consisting of a joint strategy and a final outcome) can be used, under certain conditions, as proposals and counter proposals. In fact, a coalition need only consider the potential of its actions to lead to some alternative in order to use that alternative as a proposal, but not necessarily whether other players will be willing to go along with the corresponding, to such alternative, actions. Stability is obtained if there is not a counter proposal to a given alternative proposed for adoption.

In a game with conditional strategies, the α -core is the core-concept that most closely resembles the usual core. In fact, for a game with independent strategies the α -core is identical to the usual core.

A comparison of the core^{*} and the α -core reveals that, in the absence of non-coalition-specific externalities in an economy, the two could be identical to each other. However, and in the presence of a non-coalition-specific externality, such as, the beneficial externality that could result from the production of a pure public good by a coalition, the two may be as diverse as to have the core^{*} empty while the α -core is very large.

Our interest to solution concepts for cooperative games does not end in Chapter Three. In Chapter Four, motivated from the fact that a number of solution concepts to n -person cooperative games have been defined, in various works, on the premise that, in a given game, a payoff vector can be considered stable if it offers security to the players, we use such criterion to introduce the sets of individually strongly defensible positions and individually defensible positions, as new solution concepts to cooperative games in characteristic function form, with transferable utility, and side payments.

In general terms, a payoff vector is an individually defensible position for a cooperative game if it offers security to the players in the following sense. Each player is able, to defend his corresponding payoff against those concerted actions of the other players that, potentially, could affect him most adversely. Payoff vectors, where no such concerted actions are possible against any player, are individually strongly defensible positions for a game, and constitute a subset of the set of individually defensible positions for that game. Briefly, a payoff vector will constitute an individually strongly defensible position for a game if domination does not reduce the payoff to any player.

Subsequently, the two solution concepts, mentioned above, are used in order to advance the hypothesis that, to offer security to the players may not be sufficient to induce stability of a payoff vector, and that an additional criterion, a maximum benefit criterion such as Pareto optimality, may be necessary. It is then proved that the core of a game satisfies this hypothesis by showing that the Pareto optimal, individually strongly defensible positions, of a game constitute the core of that game. The solution concept that emerges from the above hypothesis, if individually defensible positions are combined with Pareto optimality, is the effective core of a game. It is then proved that the effective core of each classical game is not empty and that the kernel and the bargaining set $M_1^{(1)}$ of a game may contain Pareto optimal payoff vectors which are not individually defensible positions for that game.

In Chapter Five, the solution concepts introduced in Chapter Four are applied to the cooperative games introduced in Chapter Two, relative to the notions of dominance introduced in Chapter Three. Thus in Chapter Five, by combining material from Chapters Two, Three, and Four, we lay the foundations of a theory for individually defensible allocations in an economy. The first result that emerges from such theory is that the Lindahl equilibrium allocations of an economy with crowded public goods may not be individually defensible.

6.2 Concluding Remarks

The two cooperative games introduced in Chapter Two, and the two concrete core-concepts introduced in Chapter Three, do not necessarily exhaust all possibilities, for cooperative games with respect to a given economic environment $\underline{e}$, and core-concepts for such games. In particular, and due to the definition of the abstract notion of dominance

in Section 3.2, and the abstract domination relationship that it induces (Definition 3.1), other core-concepts can be defined in between the core^{*} and the α -core. Of course this will require that we specify some new conditions under which a coalition "can be assured of an alternative."

The use of a cooperative game $\Gamma(\underline{e})$ to represent conflicts of interest in an economy $\underline{e}$ may be considered a breakthrough in cooperative game theory. However, it should be noted that our ability to define such cooperative game is a result of the particular way that we have modeled an economic environment $\underline{e}$. Thus, and as an abstract cooperative game, the game $\Gamma(\underline{e})$ will make sense only if we could introduce concepts that will be equivalent to what we have used as coalition actions with respect to different activities.

That strategies for coalitions may have a conditionality feature was pointed out by Rosenthal [32], as we have noted in the Introduction, who introduced cooperative games in effectiveness form, as he calls them. However, the resemblance of the conditionality feature of strategies for coalitions in a game $\Gamma(\underline{e})$ and that of a game in effectiveness form ends here, because Rosenthal does not consider the possibility that the members of any given coalition could also be members of a different coalition, relative to different activities. It is this latter characteristic that makes coalition actions conditional strategies, in the sense that we use it here, for a game $\Gamma(\underline{e})$, and this in extension may include the conditionality feature discussed in Rosenthal [32], in the case of an economy with externalities.

Individually defensible positions defined for an economy which can be represented by a classical game, will, apparently, imply that such an economy will always have a non-empty effective core. Thus the effective

core can be used as a solution concept that could guide us as to what allocations are stable in such economies if externalities are present, and if, due to such externalities, the core is empty as it was pointed out by Shapley and Shubik [39].

With respect to the effective core of a game one could pursue further the relationships discussed in Section 4.6, with other solution concepts, such as the nucleolus of a game.

The result obtained in Section 5.3, is a small contribution to the theory for individually defensible allocations in an economy. As we have pointed out in Section 5.1, only the foundations of such theory were laid in Chapter Five. Full development of that theory and its implications could address questions of existence for the effective core of a game $\Gamma(\underline{e})$, theories of the firm, social choice problems, and so on.

APPENDIX
THE EFFECTIVE CORE FOR 3-PERSON GAMES Γ

In Chapter Four we showed that every game $\Gamma = (N, v)$ has a non-empty effective core. It may be possible then to derive a general formula for calculating the effective core of such games. Below we do so for every 3-person game $\Gamma = (N, v)$.

For simplicity let us write the important information concerning any such game as follows:

$$N = \{1, 2, 3\}$$

$$v(\{1, 2\}) = a$$

$$v(\{1, 3\}) = b$$

$$v(\{2, 3\}) = c$$

$$x(N) = d, \text{ any } x \in X^*(\Gamma), X^*(\Gamma) \text{ the set of Pareto optimal payoff vectors of } \Gamma.$$

Three important facts that hold true for any such game are the following.

1. Pareto optimal payoff vectors are dominated, if at all, only by 2-person coalitions, i.e., for any $x \in X^*(\Gamma)$ such that $C_\Gamma(dx) \neq \emptyset$, what holds true is that $\tilde{c} \in C_\Gamma(dx)$ implies that $\tilde{c} = \{i, j\}$, $i, j \in N$, $i \neq j$.

2. The set of Pareto optimal payoff vectors such that $x(\tilde{c}) \geq v(\tilde{c})$, for any $\tilde{c} = \{i, j\}$, $i, j \in N$, $i \neq j$, is a convex and compact set. Furthermore, for any payoff vector x in such set such that $x(\tilde{c}) = v(\tilde{c})$, $\tilde{c}$ as above, what holds true is that $x^k = d - v(\tilde{c})$, for $k \notin \tilde{c}$.

3. $K(\Gamma) = \emptyset$ if, and only if, $a + b + c > 2d$. For example, if, $a + b + c > 2d$, and $K(\Gamma) \neq \emptyset$, then there exists $x \in X^*(\Gamma)$, such that $x^i + x^j \geq v(\{i, j\})$ for all coalitions $\{i, j\}$, $i, j \in N$, $i \neq j$. Summing over all such coalitions we will obtain $2d \geq a + b + c$, a contradiction. For the converse, observe that if $K(\Gamma) = \emptyset$ then, $v(\{i, j\}) \geq v(\{i\}) + v(\{j\})$, for all coalitions $\{i, j\}$, $i, j \in N$, $i \neq j$. Otherwise, and if for some $\{i, j\}$, $v(\{i, j\}) < v(\{i\}) + v(\{j\})$, the payoff vector $\bar{x}$, $\bar{x} \in X^*(\Gamma)$, such that $x^i = v(\{i\})$, $x^j = v(\{j\})$, and $x^k = \text{Max}\{(d - x^i - x^j), (v(\{i, k\}) - x^i), (v(\{j, k\}) - x^j)\}$, for $k \neq i \neq j$, will not be dominated by any coalition. I.e., in such case $K(\Gamma) \neq \emptyset$, a contradiction. But then, for any payoff vector $\bar{x} \in X^*(\Gamma)$, either $a + b + c > 2x(N) = 2d$, and the proof follows, or $2d = 2x(N) \geq a + b + c$, and we can find x^* such that $x^* \in X^*(\Gamma)$, and $x^* \in K(\Gamma)$ which is a contradiction.

With the above facts in mind, and due to Theorem 4.1 in Chapter Four, if $SD^{*i}(\Gamma) \equiv (X^*(\Gamma) \cap SD^i(\Gamma))$, then,

$$(A.1) \quad SD^{*i}(\Gamma) = \{x \in X^*(\Gamma) : x(\{j, k\}) \geq v(\{j, k\}), \\ j, k \in N, j \neq k \neq i\}, \text{ any } i \in N.$$

By 2, above, if x_L^{*i} denotes the maximum payoff to player i , in any $x \in SD^{*i}(\Gamma)$, i.e., if $x_L^{*i} \equiv \text{Max}\{p^i(x) : x \in SD^{*i}(\Gamma)\}$, then,

$$(A.2) \quad x_L^{*1} = (d-c), x_L^{*2} = (d-b), \text{ and } x_L^{*3} = (d-a),$$

and any $x, x \in X^*(\Gamma)$, such that $x_L^{*i} \geq x^i$, is such that $x \in SD^{*i}(\Gamma)$.

By Theorem 4.3, in Chapter Four, for any game $\Gamma = (N, v)$, $K(\Gamma) = (X^*(\Gamma) \cap SD(\Gamma))$. On the other hand the set $F(\Gamma) = X(\Gamma) - \bigcup_{i \in N} SD^i(\Gamma)$. Therefore, for any 3-person game $\Gamma = (N, v)$, such that $K(\Gamma) \neq \emptyset$, what holds

true is that $(X^*(\Gamma) \cap F(\Gamma)) = \emptyset$, and therefore $EK(\Gamma) = K(\Gamma)$. (Note that as we have seen in Section 4.5, Chapter Four, this is not necessarily true for games $\Gamma = (N, v)$ with more than three players.)

The above conclusion then says that the effective core of every 3-person game is identical to the core of that game provided that the core is not empty.

Let us suppose now that for a given game $\Gamma = (N, v)$, as above, $K(\Gamma) = \emptyset$.

From (A.2), above, it is easy to verify that for any 2-person coalition $\{i, j\}$, $i \neq j$, the following holds true.

$$(A.3) \quad (v(\{i, j\}) - (x_L^{*i} + x_L^{*j})) = (a + b + c - 2d).$$

Note that the right-hand-side of (A.3), is constant, hence, independent of the particular coalition $\{i, j\}$, and positive if $K(\Gamma) = \emptyset$.

Let $D^{*i}(\Gamma) \equiv (X^*(\Gamma) \cap SD^i(\Gamma))$, for each $i \in N$, let

$$(A.4) \quad x_U^{*i} \equiv \text{Max}\{p^i(x) : x \in D^{*i}(\Gamma)\},$$

and let

$$(A.5) \quad x^{*i} \equiv x_L^{*i} + \frac{a + b + c - 2d}{2}.$$

We claim that $x_U^{*i} = x^{*i}$, $\forall i \in N$, and therefore that

$$(A.6) \quad x_U^{*1} = \frac{a+b-c}{2}, \quad x_U^{*2} = \frac{a+c-b}{2}, \quad \text{and} \quad x_U^{*3} = \frac{b+c-a}{2}.$$

It is easy to verify that $x^{*i} + x^{*j} = v(\{i, j\})$, and that $x^{*i} + x^{*j} + x_L^{*k} = d$, for any choice of $i, j, k \in N$, $i \neq j \neq k$. Let us consider now the Pareto optimal payoff vector, $\bar{x}$, $\bar{x} = (x^{*i}, x^{*j}, x_L^{*k})$. To show that $\bar{x} \in D^{*i}(\Gamma)$. Suppose that $\bar{x} \notin D^{*i}(\Gamma)$. Then $EB_\Gamma^i(\bar{x}) \neq \emptyset$. Hence, there exists an i.r.p.c. $(\tilde{x}; \tilde{\tau}) \in EB_\Gamma^i(\bar{x})$, such that $\tilde{\tau} = \{\{j, k\}, \{i\}\}$,

$\tilde{x}(\{j, k\}) = v(\{j, k\})$, and $\tilde{x} \in \underline{\text{dom}}_{\{j, k\}} \bar{x}$. But in any such vector

$$(A.7) \quad x_L^{*k} \leq \tilde{x}^k \leq x_L^{*k} + \frac{a + b + c - 2d}{2} = x^{*k},$$

and $(x^{*i} + x^{*k}) = v(\{i, k\})$. Therefore, player i can form a coalition with player k and regain his original payoff x^{*i} , a contradiction, since the standing hypothesis is that $\bar{x} \notin D^{*i}(\Gamma)$. Note that the same argument can be made with respect to player j .

Having proved that vectors of the form: $\bar{x} = (x^{*i}, x^{*j}, x_L^{*k})$ belong to the set $D^{*i}(\Gamma)$, let us suppose that such vectors do not solve the maximization problem implicit in (A.4). A pairwise comparison of players will disprove such supposition. In other words, and because a vector $\bar{x}$, as above is such that $\bar{x} \in D^{*i}(\Gamma) \cap D^{*j}(\Gamma)$, if any one of the players i or j was to receive a higher payoff than x^{*i} or x^{*j} , then the contradiction established above will not hold, i.e., the player s , $s \in \{i, j\}$, who receives a higher payoff than x^{*s} , will not be able to regain his original payoff.

Theorem: For any 3-person game $\Gamma = (N, v)$ such that $K(\Gamma) = \emptyset$, the set $EK(\Gamma)$ is such that $EK(\Gamma) = \{x \in X^*(\Gamma) : ((d-c), (d-b), (d-a)) \leq x \leq ((\frac{a+b-c}{2}), (\frac{a+c-b}{2}), (\frac{b+c-a}{2}))\}$.

Proof: Let us observe that neither one of the weak inequalities $\leq$, in the above set, can hold as an equality due to the fact that $K(\Gamma) = \emptyset$. Suppose then that there exists a payoff vector $\bar{x}$ that belongs to the set on the right-hand-side of the equality in the theorem but $\bar{x} \notin EK(\Gamma)$. Then, $\bar{x} \notin \bigcap_{i \in N} D^{*i}(\Gamma)$. Therefore, there exists a player $i \in N$, such that $\bar{x} \notin D^{*i}(\Gamma)$. Without loss of generality let us suppose that this is player 1. This means $(d-c) < \bar{x}^1$, otherwise $\bar{x} \in SD^{*1}(\Gamma)$, $\bar{x}^1 \leq \frac{a+b-c}{2}$,

and $EB_{\Gamma}^1(\bar{x}) \neq \emptyset$. A similar argument to that used to prove (A.6) is sufficient to show that this will lead to a contradiction.

To prove the converse, let us suppose that there exists a payoff vector $\bar{x}$, $\bar{x} \in EK(\Gamma)$, and such payoff vector has not been included on the right-hand-side of the equality in the theorem. Apparently, such payoff vector must be Pareto optimal, belongs to the set $D(\Gamma)$, and violates one of the inequalities on the right-hand-side of the equation in the theorem. Because we have already excluded the possibility that neither one of the weak inequalities can hold as a strict equality, and due to (A.6), what must be true for such payoff vector is that $\bar{x}^i < x_L^{*i}$, for some $i \in N$. But then, $\bar{x}(\{j, k\}) > v(\{j, k\})$, for $j \neq k \neq i$, and $\bar{x} \notin D^j(\Gamma) \cap D^k(\Gamma)$, because as we have seen $x_U^{*j} + x_U^{*k} = v(\{j, k\})$, a contradiction.

q.e.d.

The 3-dimensional figure on page 168, below, demonstrates some of the major points discussed in this Appendix. The area OABC represents the set of payoff vectors $X(\Gamma)$ of a 3-person game with an empty core.

The three axes x^1 , x^2 , and x^3 represent payoffs to players 1, 2, and 3, respectively.

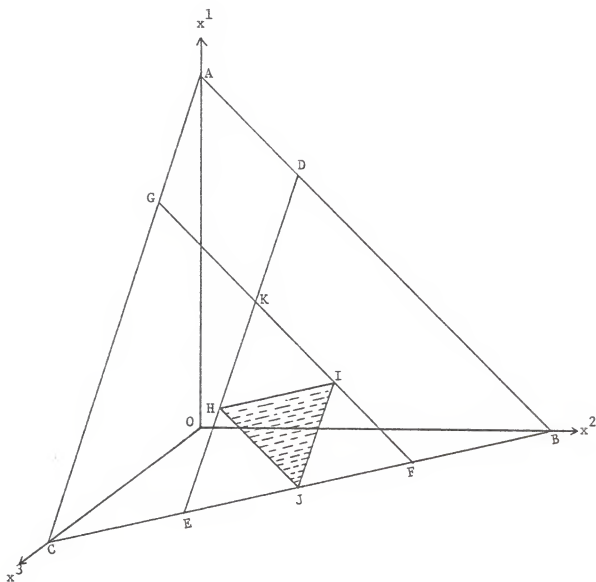
The point O represents the payoff vector $(v(\{1\}), v(\{2\}), v(\{3\}))$ and is taken as the origin.

The area ABC represents the Pareto optimal points $X^*(\Gamma)$ of the game. Thus, the lines GF, DE, and CB represent the Pareto optimal payoff vectors where each coalition $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$, respectively, receives exactly its value. As a consequence, CB, ADEC, and ABFG, represent the Pareto optimal points of the sets of strongly defensible positions for players 1, 2, and 3, respectively, i.e., they represent the sets

$SD^{*1}(\Gamma)$, $SD^{*2}(\Gamma)$, and $SD^{*3}(\Gamma)$, respectively. Observe that these sets have an empty intersection, and thus, the core of this game is empty.

The set $(\overline{F}(\Gamma) \cap X^*(\Gamma))$ is represented in the Figure by the area KFE.

The Pareto optimal points of the sets of defensible positions for each player in the above game, i.e., the sets $D^{*1}(\Gamma)$, $D^{*2}(\Gamma)$, and $D^{*3}(\Gamma)$, are represented in the Figure by CEHIFB, ADKIJEC, and AGKHJFB, respectively, and they have a non-empty intersection consisting of the shaded area in the Figure, HIJ. Therefore, HIJ represents the effective core of this game.



FIGURE

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BIOGRAPHICAL SKETCH

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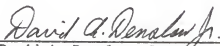
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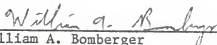
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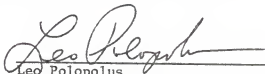
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